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▶ To cite this version:

Vuk Milisic, Ulrich Razafison. Weighted Sobolev spaces for the Laplace equation in periodic infinite strips. 2014. hal-00728408v3

HAL Id: hal-00728408
https://hal.science/hal-00728408v3

Preprint submitted on 6 Jun 2014

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Weighted Sobolev spaces for the Laplace equation in periodic infinite strips

Vuk Milišić[*] Ulrich Razafison[†]

Abstract

This paper establishes isomorphisms for the Laplace operator in weighted Sobolev spaces (WSS). These H^m_α-spaces are similar to standard Sobolev spaces $H^m(\mathbb{R}^n)$, but they are endowed with weights $(1+|x|^2)^{\alpha/2}$ prescribing functions' growth or decay at infinity. Although well established in $\mathbb{R}^n$ [1], these weighted results do not apply in the specific hypothesis of periodicity. This kind of problem appears when studying singularly perturbed domains (roughness, sieves, porous media, etc) : when zooming on a single perturbation pattern, one often ends with a periodic problem set on an infinite strip. We present a unified framework that enables a systematic treatment of such problems. We provide existence and uniqueness of solutions in our WSS. This gives a refined description of solution's behavior at infinity which is of importance in the mutli-scale context. We then identify these solutions with the convolution of a Green function (specific to infinite strips with periodic boundary conditions) and the given data. These identification and isomorphisms are valid for any real α and any integer m out of a countable set of critical values. They require polar conditions on the data which are often not satisfied in the homogenization context, in this case as well, we construct a solution and provide refined weighted estimates.

Keywords:

AMS Classification: Primary 47B37; Secondary 35C15, 42B20, 31A10

1 Introduction

In this article, we solve the Laplace equation in a 1-periodic infinite strip in two space dimensions:

$$\Delta u = f, \quad \text{in } Z :=]0,1[\times\mathbb{R}. \tag{1.1}$$

As the domain is infinite in the vertical direction, one introduces weighted Sobolev spaces describing the behavior at infinity of the solution. This behavior is related to weighted Sobolev properties of f.

The usual weights, when adapted to our problem, are polynomial functions at infinity and regular bounded functions in the neighborhood of the origin: they are powers of $\rho(y_2) := (1+y_2^2)^{1/2}$ and, in some critical cases, higher order derivatives are completed by logarithmic functions ($\rho(y_2)^\alpha \log^\beta(1+\rho(y_2)^2)$).

[*] Université Paris 13, Laboratoire Analyse, Géométrie et Applications, CNRS UMR 7539, France (milisic@math.univ-paris13.fr)

[†] Laboratoire de Mathématiques, CNRS UMR 6623, Université de Franche-Comté, 16 route de Gray 25030 Besançon Cedex,France (ulrich.razafison@univ-fcomte.fr)

The literature on the weighted Sobolev spaces is wide [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and deals with various types of domains. To our knowledge, this type of weights has not been applied to problem (1.1). The choice of the physical domain comes from periodic singular problems : in [12, 13, 14], a zoom around a domain's periodic ϵ-perturbation leads to set an obstacle of size 1 in Z and to consider a microscopic problem defined on *a boundary layer*. The behavior at infinity of this *microscopic* solution is of importance: it provides an averaged feed-back on the macroscopic scale (see [15] and references therein). This paper presents a systematic analysis of such microscopic problems. We intend to give a standard framework to skip tedious and particular proofs related to the unboundedness of Z.

We provide isomorphisms of the Laplace operator between our weighted Sobolev spaces. It is the first step among results in the spirit of [3, 1, 16]. Since error estimates for boundary layer problems [12, 14, 13] are mostly performed in the H^s framework we focus here on weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(Z)$ with $p = 2$. There are three types of tools used : arguments specifically related to weighted Sobolev spaces [1, 17], variational techniques from the homogenization literature [14, 18, 19] and some potential theory methods [20]. A general scheme might illustrate how these ideas relate one to each other :

- a Green function G specific to the periodic infinite strip is exhibited for the Laplace operator. The convolution of f with G provides an explicit solution to (1.1). A particular attention is provided to give the weakest possible meaning to the latter convolution under minimal requirements on f.
- variational *inf-sup* techniques specially adapted to the weighted spaces provide existence and uniqueness theorems for a restricted range of weights. This leads to first series of isomorphisms results in the variational context.
- these arguments are then applied to weighted derivatives and give natural regularity shift results in $H^m_{\alpha,\#}(Z)$ for $m \geq 2$ (see below).
- by duality and appropriate use of generalized Poincaré estimates (leading to interactions - orthogonality or quotient spaces - with various polynomial families), one ends with generic isomorphism results that read

Theorem 1.1. *For any $m \in \mathbb{Z}$, for any $\alpha \in \mathbb{R}$ such that $\alpha \notin \mathbb{Z} + \frac{1}{2}$, the mapping*

$$\Delta : H^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]} \mapsto H^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[-m-1/2+\alpha]}$$

is an isomorphism.

The spaces $\mathbb{P}'^{\Delta}_{[m-1/2-\alpha]}$ of harmonic polynomials included in $H^m_{\alpha,\#}(Z)$ are defined in section 2. The previous theorem states that if one looks for a solution u of (1.1) that decays fast enough at infinity, then uniqueness is insured but the counter part is that the datum f must satisfy a polar condition. On the contrary, if the previous condition on u is released, then uniqueness is obtained up to harmonical polynomials and the datum f do not have to satisfy any polar condition. These properties are well-known when studying elliptic equations in weighted Sobolev spaces (see for instance [1]).

- for all values of $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}$ such that $\alpha \notin \mathbb{Z} + \frac{1}{2}$, we identify explicit solutions obtained via convolution with solutions given in Theorem 1.1. This gives our second main result:

  Theorem 1.2. *Let $m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ such that $\alpha \notin \mathbb{Z} + \frac{1}{2}$, and $f \in H^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q[-m-1/2-\alpha]}$. Then $G * f \in H^{m+2}_{\alpha,\#}(Z)$ is the unique solution of the Laplace equation (5.20) up to a polynomial of $\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]}$. Moreover, we have the estimate*

$$\|G * f\|_{H^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]}} \leq C\|f\|_{H^m_{\alpha,\#}(Z)}.$$

- For $\alpha > m+1/2$, $\alpha \notin \mathbb{Z}+\frac{1}{2}$, when the datum $f \in H^m_{\alpha,\#}(Z)$ does not satisfy the polar condition $f \perp \mathbb{P}'^{\Delta}_{[-m-1/2+\alpha]}$, we set up an appropriate decomposition of f, and construct the solution using previous results. A singular behavior of this solution at infinity is identified : it behaves like a linear combination of $\operatorname{sgn}(y_2)$ and $|y_2|$. As for corner domains [21], when the singular part is removed, one recovers the weighted regularity estimates. This provides Theorem 7.1. This result is of particular interest in the context of homogenization.

The paper is organized as follows. In section 2, we define the basic functional framework and some preliminary results. In section 3, we adapt weighted Poincaré estimates to our setting. Then (section 4), we introduce a mixed Fourier transform (MFT) : it is a discrete Fourier transform in the horizontal direction and a continuous transform in the vertical direction. At this stage, we prove a series of isomorphisms in the non-critical case (section 5) by variational techniques. The MFT operator allows an explicit computation of a Green function (section 6). We show, as well, weighted and standard estimates of the convolution with the latter fundamental solution. Then we identify any of the solutions above with the convolution between G and the data f, this proves Theorem 1.2. Finally, when the polar conditions is not satisfied by the data, we construct a solution in section 7. In the appendices, we provide either technical proofs of some of the claims of the paper (see Appendices A, B, C, D and F), or results that we did not find in the literature but that are of interest in this context (Appendix E).

2 Notation, preliminary results and functional framework

2.1 Notation and preliminaries

We denote by Z the two-dimensional infinite strip defined by $Z =]0,1[\times\mathbb{R}$. We use bold characters for vector or matrix fields. A point in $\mathbb{R}^2$ is denoted by $\boldsymbol{y} = (y_1, y_2)$ and its distance to the origin by $r = |\boldsymbol{y}| = \left(y_1^2 + y_2^2\right)^{1/2}$. Let $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{Z}$ the set of all integers and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. We denote by $[k]$ the integer part of k. For any $j \in \mathbb{Z}$, $\mathbb{P}'_j$ stands for the polynomial space of degree less than or equal to j that only depends on y_2. If j is a negative integer, we set by convention $\mathbb{P}'_j = \{0\}$. We define $\mathbb{P}'^{\Delta}_j$ the subspace of harmonic polynomials of $\mathbb{P}'_j$. The support of a function φ is denoted by $\operatorname{supp}(\varphi)$. We recall that $\mathscr{D}(\mathbb{R})$ and $\mathscr{D}(\mathbb{R}^2)$ are spaces of $\mathscr{C}^\infty$ functions with compact support in $\mathbb{R}$ and $\mathbb{R}^2$ respectively, $\mathscr{D}'(\mathbb{R})$ and $\mathscr{D}'(\mathbb{R}^2)$ their dual spaces, namely the spaces of distributions. We denote by $\mathscr{S}(\mathbb{R})$ the Schwartz space of functions in $\mathscr{C}^\infty(\mathbb{R})$ with rapid decrease at infinity, and by $\mathscr{S}'(\mathbb{R})$ its dual, i.e. the space of tempered distributions. We recall that, for $m \in \mathbb{N}$, H^m is the classical Sobolev space and we denote by $H^m_\#(Z)$ the space of functions that belong to $H^m(Z)$ and that are 1-periodic in the y_1 direction. Given a Banach space B with its dual B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, *i.e.*:

$$B' \perp X = \{f \in B',\ \forall v \in X, \langle f, v\rangle = 0\} = (B/X)'.$$

We introduce τ_λ the operator of translation of $\lambda \in \mathbb{Z}$ in the y_1 direction. If $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a function, then we have

$$(\tau_\lambda \Phi)(\boldsymbol{y}) := \Phi(y_1 - \lambda, y_2).$$

For any $\Phi \in \mathscr{D}(\mathbb{R}^2)$, we set $\overline{\omega}\Phi := \sum_{\lambda \in \mathbb{Z}} \tau_\lambda \Phi$, which is the y_1-periodical transform of Φ. The mapping $y_1 \mapsto \overline{\omega}\Phi(\boldsymbol{y})$ belongs to $\mathscr{C}^\infty(\mathbb{R})$ and is 1-periodic. Observe that there exists a function θ satisfying

$$\theta \in \mathscr{D}(\mathbb{R}) \text{ and } \overline{\omega}\theta = 1.$$

More precisely, consider a function $\psi \in \mathscr{D}(\mathbb{R})$ such that $\psi > 0$ on the interior of its support. Then we simply set $\theta := \frac{\psi}{\overline{\omega}\psi}$. The function θ is called a periodical $\mathscr{D}(\mathbb{R})$-partition of unity.

If $T \in \mathscr{D}'(\mathbb{R}^2)$, then for all $\varphi \in \mathscr{D}(\mathbb{R}^2)$, we set

$$\langle \tau_\lambda T, \varphi\rangle := \langle T, \tau_{-\lambda}\varphi\rangle,\ \ \lambda \in \mathbb{Z}.$$

Similarly, if $T \in \mathscr{D}'(\mathbb{R}^2)$ has a compact support in the y_1 direction, then the y_1-periodical transform of T, denoted by $\overline{\omega}T$, is defined by

$$\langle \overline{\omega}T, \varphi\rangle := \langle T, \overline{\omega}\varphi\rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R}^2).$$

These definitions are well-known, we refer for instance to [22] and [23].

Remark 2.1. Let Φ be in $\mathscr{D}(\mathbb{R}^2)$. Then we have

$$\tau_\lambda \partial_i \Phi = \partial_i(\tau_\lambda \Phi), \quad \forall \lambda \in \mathbb{Z}, \tag{2.2}$$

$$\overline{\omega}(\partial_i \Phi) = \partial_i(\overline{\omega}\Phi) \tag{2.3}$$

and

$$\|\tau_\lambda \Phi\|_{L^2(Z)} \leq \|\Phi\|_{L^2(\mathbb{R}^2)}. \tag{2.4}$$

As a consequence of equality (2.3), if $u \in \mathscr{D}'(\mathbb{R}^2)$, we also have

$$\overline{\omega}(\partial_i u) = \partial_i(\overline{\omega}u). \tag{2.5}$$

The next lemma is used to prove the density result of Proposition 2.3.

Lemma 2.1. *Let K be a compact of $\mathbb{R}^2$. Let u be in $H^m(\mathbb{R}^2)$ and have a compact support included in K. Then we have*

$$\|\overline{\omega}\, u\|_{H^m(Z)} \leq N(K)\|u\|_{H^m(\mathbb{R}^2)},$$

where $N(K)$ is an integer only depending on K.

Proof. Let us first notice that, since K is compact, there is a finite number of $\lambda \in \mathbb{Z}$ such that $\operatorname{supp}(\tau_\lambda u) \cap [0,1] \times \mathbb{R}$ is not an empty set. This number is bounded by a finite integer $N(K)$ that only depends on K. It follows that $\overline{\omega}\, u$ is a finite sum and

$$\|\overline{\omega}\, u\|_{L^2(Z)} \leq N(K)\|u\|_{L^2(\mathbb{R}^2)}.$$

The end of the proof then follows from (2.5). □

We now define the following spaces

$$\begin{aligned}
\mathscr{C}^\infty_\#(Z) = \{ \quad &\varphi : Z \mapsto \mathbb{R},\ y_1 \mapsto \varphi(\boldsymbol{y}) \in \mathscr{C}^\infty([0,1]) \text{ 1-periodic},\ y_2 \mapsto \varphi(\boldsymbol{y}) \in \mathscr{C}^\infty(\mathbb{R})\},\\
\mathscr{D}_\#(Z) = \{ \quad &\varphi : Z \mapsto \mathbb{R},\ y_1 \mapsto \varphi(\boldsymbol{y}) \in \mathscr{C}^\infty([0,1]) \text{ 1-periodic},\ y_2 \mapsto \varphi(\boldsymbol{y}) \in \mathscr{D}(\mathbb{R})\},\\
\mathscr{S}_\#(Z) = \{ \quad &\varphi : Z \mapsto \mathbb{R},\ y_1 \mapsto \varphi(\boldsymbol{y}) \in \mathscr{C}^\infty([0,1]) \text{ 1-periodic},\ y_2 \mapsto \varphi(\boldsymbol{y}) \in \mathscr{S}(\mathbb{R})\}.
\end{aligned}$$

The dual spaces of $\mathscr{D}_\#(Z)$ and $\mathscr{S}_\#(Z)$ are denoted by $\mathscr{D}'_\#(Z)$ and $\mathscr{S}'_\#(Z)$ respectively.

2.2 Weighted Sobolev spaces in an infinite strip

We introduce the weight $\rho = \rho(y_2) := \left(1 + y_2^2\right)^{1/2}$. Observe that, for $\lambda \in \mathbb{N}$ and for $\gamma \in \mathbb{R}$, as $|y_2|$ tends to infinity, we have

$$\left|\frac{\partial^\lambda \rho^\gamma}{\partial y_2^\lambda}\right| \leq C\rho^{\gamma-\lambda}. \tag{2.6}$$

For $\alpha \in \mathbb{R}$, we define the weighted space

$$L^2_\alpha(Z) := H^0_{\alpha,\#}(Z) = \{u \in \mathscr{D}'(Z),\ \rho^\alpha u \in L^2(Z)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{L^2_\alpha(Z)} = \|\rho^\alpha u\|_{L^2(Z)}.$$

Proposition 2.1. *The space $\mathscr{D}(Z)$ is dense in $L^2_\alpha(Z)$.*

Proof. Let u be in $L^2_\alpha(Z)$. Then, by definition of the space $L^2_\alpha(Z)$, it follows that $\rho^\alpha u \in L^2(Z)$. Therefore there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathscr{D}(Z)$ such that v_n converges to $\rho^\alpha u$ in $L^2(Z)$ as $n \to \infty$. Thus, setting $u_n = \rho^{-\alpha} v_n$, we see that u_n converges to u in $L^2_\alpha(Z)$ as $n \to \infty$. □

Remark 2.2. Observe that we have the algebraic inclusion $\mathscr{D}(Z) \subset \mathscr{D}_\#(Z)$. It follows that the space $\mathscr{D}_\#(Z)$ is dense in $L^2_\alpha(Z)$.

For a non-negative integer m and a real number α, we set

$$k = k(m,\alpha) = \begin{cases} -1 & \text{if} \quad \alpha \notin \{1/2,\dots,m-1/2\} \\ m-1/2-\alpha & \text{if} \quad \alpha \in \{1/2,\dots,m-1/2\} \end{cases}$$

and we introduce the weighted Sobolev space

$$\begin{aligned} H^m_{\alpha,\#}(Z) = \{u \in \mathscr{D}'_\#(Z); \forall \boldsymbol{\lambda} \in \mathbb{N}^2, 0 \le |\boldsymbol{\lambda}| \le k,\ \rho^{-m+|\boldsymbol{\lambda}|}(\ln(1+\rho^2))^{-1}\partial^{\boldsymbol{\lambda}} u \in L^2_\alpha(Z), \\ k+1 \le |\boldsymbol{\lambda}| \le m,\ \rho^{-m+|\boldsymbol{\lambda}|}\partial^{\boldsymbol{\lambda}} u \in L^2_\alpha(Z)\}, \end{aligned}$$

which is a Banach space when endowed with the norm

$$\begin{aligned} \|u\|_{H^m_{\alpha,\#}(Z)} = \Bigg(\sum_{0\le|\boldsymbol{\lambda}|\le k} \|\rho^{-m+|\boldsymbol{\lambda}|}(\ln(1+\rho^2))^{-1}\partial^{\boldsymbol{\lambda}} u\|^2_{L^2_\alpha(Z)} \\ + \sum_{k+1\le|\boldsymbol{\lambda}|\le m} \|\rho^{-m+|\boldsymbol{\lambda}|}\partial^{\boldsymbol{\lambda}} u\|^2_{L^2_\alpha(Z)} \Bigg)^{1/2}. \end{aligned} \tag{2.7}$$

We define the semi-norm

$$|u|_{H^m_{\alpha,\#}(Z)} = \left(\sum_{|\boldsymbol{\lambda}|=m} \|\partial^{\boldsymbol{\lambda}} u\|^2_{L^2_\alpha(Z)} \right)^{1/2}.$$

Let us give an example of such weighted space. Consider the case $m=1$, then if $\alpha \neq 1/2$, we have

$$H^1_{\alpha,\#}(Z) = \{u \in \mathscr{D}'_\#(Z); u \in L^2_{\alpha-1}(Z), \nabla u \in L^2_\alpha(Z)\}$$

and

$$H^1_{1/2,\#}(Z) = \{u \in \mathscr{D}'_\#(Z); \ln(1+\rho^2)^{-1} u \in L^2_{\alpha-1}(Z), \nabla u \in L^2_\alpha(Z)\}. \tag{2.8}$$

Observe that the logarithmic weight function only appears for the so-called critical cases $\alpha \in \{1/2,\dots,m-1/2\}$. Local properties of the spaces $H^m_{\alpha,\#}(Z)$ coincide with those of the classical Sobolev space $H^m_\#(Z)$. When $\alpha \notin \{1/2,\dots,m-1/2\}$ we have the following algebraic and topological inclusions :

$$H^m_{\alpha,\#}(Z) \subset H^{m-1}_{\alpha-1,\#}(Z) \subset \dots \subset L^2_{\alpha-m}(Z). \tag{2.9}$$

When $\alpha \in \{1/2,\dots,m-1/2\}$, the logarithmic weight functions appear, so that we only have the inclusions

$$H^m_{\alpha,\#}(Z) \subset \dots \subset H^{m-\alpha+1/2}_{1/2,\#}(Z). \tag{2.10}$$

Observe that the mapping

$$u \in H^m_{\alpha,\#}(Z) \mapsto \partial^{\boldsymbol{\lambda}} u \in H^{m-|\boldsymbol{\lambda}|}_{\alpha,\#}(Z) \tag{2.11}$$

is continuous for $\boldsymbol{\lambda} \in \mathbb{N}^2$. Using (2.6), for $\alpha, \gamma \in \mathbb{R}$ such that $\alpha \notin \{1/2,\dots,m-1/2\}$ and $\alpha-\gamma \notin \{1/2,\dots,m-1/2\}$, the mapping

$$u \in H^m_{\alpha,\#}(Z) \mapsto \rho^\gamma u \in H^m_{\alpha-\gamma,\#}(Z) \tag{2.12}$$

is an isomorphism.

Proposition 2.2. *Let q be the greatest non-negative integer such that $y_2^q \in H^m_{\alpha,\#}(Z)$. Then q reads :*

- *if $m \in \mathbb{N}^*$*

$$q = q(m,\alpha) := \begin{cases} m - 3/2 - \alpha, & \text{if } \alpha + 1/2 \in \{i \in \mathbb{Z}; i \le 0\}, \\ [m - 1/2 - \alpha], & \text{otherwise.} \end{cases} \tag{2.13}$$

- *if $m = 0$ one has :*

$$q = q(0,\alpha) := \begin{cases} -3/2 - \alpha, & \text{if } \alpha + 1/2 \in \mathbb{Z}, \\ [-1/2 - \alpha], & \text{otherwise.} \end{cases} \tag{2.14}$$

In the table below, we give some of polynomials belonging to $\mathbb{P}_{q(m,\alpha)}$ for different values of m and for α in the range $[-\frac{5}{2}, \frac{5}{2}]$.

$m \setminus \alpha$	$[-\frac{5}{2}; -\frac{3}{2}[$	$[-\frac{3}{2}; -\frac{1}{2}[$	$[-\frac{1}{2}; \frac{1}{2}]$	$]\frac{1}{2}; \frac{3}{2}]$	$]\frac{3}{2}; \frac{5}{2}]$
$\mathbb{P}'_{q(0,\alpha)}$	$\mathbb{P}'_1$	$\mathbb{P}'_0$	0	0	0
$\mathbb{P}'_{q(1,\alpha)}$	$\mathbb{P}'_2$	$\mathbb{P}'_1$	$\mathbb{P}'_0$	0	0
$\mathbb{P}'_{q(2,\alpha)}$	$\mathbb{P}'_3$	$\mathbb{P}'_2$	$\mathbb{P}'_1$	$\mathbb{P}'_0$	0

Table 1: Polynomial spaces included in $H^m_{\alpha,\#}(Z)$ for various values of α and m

We state now a density result whose proof is given in Appendix A.

Proposition 2.3. *The space $\mathcal{D}_\#(Z)$ is dense in $H^m_{\alpha,\#}(Z)$.*

The above proposition implies that the dual space of $H^m_{\alpha,\#}(Z)$ denoted by $H^{-m}_{-\alpha,\#}(Z)$ is a subspace of $\mathcal{D}'_\#(Z)$.

3 Weighted Poincaré estimates

Let R be a positive real number. For $\beta \neq -1$, one uses the standard Hardy estimates:

$$\int_R^\infty |f(r)|^2 r^\beta dr \le \left(\frac{2}{\beta+1}\right)^2 \int_R^\infty |f'(r)|^2 r^{\beta+2} dr, \tag{3.15}$$

while for the specific case when $\beta = -1$, one switches to

$$\int_R^\infty \frac{|f(r)|^2}{(\ln r)^2 r} dr \le \left(\frac{4}{3}\right)^2 \int_R^\infty |f'(r)|^2 r dr.$$

We now introduce the truncated domain $Z_R :=]0,1[\times (]-\infty, -R[\cup]R, +\infty[)$.
Using the above Hardy inequalities in the y_2 direction, we can easily prove the following lemma.

Lemma 3.1. *Let α, $R > 1$ be real numbers and let $m \ge 1$ be an integer. Then there exists a constant C^m_α such that*

$$\forall \varphi \in \mathcal{D}_\#(Z_R), \quad \|\varphi\|_{H^m_{\alpha,\#}(Z_R)} \le C^m_\alpha \|\partial_2 \varphi\|_{H^{m-1}_{\alpha,\#}(Z_R)}. \tag{3.16}$$

As a consequence, we also have

$$\forall \varphi \in \mathscr{D}_{\#}(Z_R), \quad \left\|\varphi\right\|_{H^m_{\alpha,\#}(Z_R)} \leq C^m_\alpha |\varphi|_{H^m_{\alpha,\#}(Z_R)}. \tag{3.17}$$

For the particular case when $\alpha \in \{1/2, ..., m-1/2\}$, (3.17) cannot hold without introducing logarithmic weights in the definition of the space $H^m_{\alpha,\#}(Z)$.
Proceeding as in [1] (Theorem 8.3), we have the Poincaré-type inequalities :

Theorem 3.1. *Let α be a real number and $m \geq 1$ an integer. Let $j := \min(q(m,\alpha), m-1)$ where $q(m,\alpha)$ is defined by Proposition 2.2. Then there exists a constant $C > 0$, such that for any $u \in H^m_{\alpha,\#}(Z)$, we have*

$$\inf_{\lambda \in \mathbb{P}'_j} \|u + \lambda\|_{H^m_{\alpha,\#}(Z)} \leq C|u|_{H^m_{\alpha,\#}(Z)}. \tag{3.18}$$

In other words, the semi-norm $|.|_{H^m_{\alpha,\#}(Z)}$ defines on $H^m_{\alpha,\#}(Z)/\mathbb{P}'_j$ a norm which is equivalent to the quotient norm.

In order to present the last results of the section, we introduce the space:

$$V_\alpha(Z) = \{\boldsymbol{v} \in L^2_\alpha(Z),\ \operatorname{div} \boldsymbol{v} = 0\}. \tag{3.19}$$

Then, as a consequence of Theorem 3.1, we have the following isomorphism results on the gradient and divergence operators, whose proof is similar to the proof of Proposition 4.1 in [1].

Proposition 3.1. *Assume that $\alpha \in \mathbb{R}$.*

1. *If $\alpha > 1/2$, then*
   - *the gradient operator is an isomorphism:*
   $$\nabla : H^1_{\alpha,\#}(Z) \mapsto L^2_\alpha(Z) \perp V_{-\alpha}(Z),$$
   - *the divergence operator is an isomorphism:*
   $$\operatorname{div} : L^2_{-\alpha}(Z)/V_{-\alpha}(Z) \mapsto H^{-1}_{-\alpha,\#}(Z).$$
2. *If $\alpha \leq 1/2$, then*
   - *the gradient operator is an isomorphism:*
   $$\nabla : H^1_{\alpha,\#}(Z)/\mathbb{R} \mapsto L^2_\alpha(Z) \perp V_{-\alpha}(Z),$$
   - *the divergence operator is an isomorphism:*
   $$\operatorname{div} : L^2_{-\alpha}(Z)/V_{-\alpha}(Z) \mapsto H^{-1}_{-\alpha,\#}(Z) \perp \mathbb{R}.$$

4 The mixed Fourier transform (MFT)

The purpose of this section is to define the MFT. We shall use this tool in the next section in order to state a general result of uniqueness of (1.1) and in section 6 in order to compute the fundamental solution.

4.1 Rapidly decreasing functions

Let set $\Gamma := \mathbb{Z} \times \mathbb{R}$ and let us write the locally convex linear topological spaces :

$$\tilde{\mathscr{S}}(\Gamma) := \Big\{ \tilde{\varphi} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R};\ \forall k \in \mathbb{Z} \quad \tilde{\varphi}(k, \cdot) \in \mathscr{S}(\mathbb{R})$$
$$\text{and} \sup_{k \in \mathbb{Z}, l \in \mathbb{R}} \left| k^{\alpha} l^{\beta} \partial^{\gamma}_{l^{\gamma}} \tilde{\varphi}(k, l) \right| < \infty, \forall (\alpha, \beta, \gamma) \in \mathbb{N}^3 \Big\}.$$

The space $\tilde{\mathscr{S}}(\Gamma)$ is endowed with the semi-norms

$$\left| \tilde{\varphi} \right|_{\alpha, \beta, \gamma} := \sup_{k \in \mathbb{Z}, l \in \mathbb{R}} \left| k^{\alpha'} l^{\beta'} \partial^{\gamma'}_{l^{\gamma'}} \tilde{\varphi}(k, l) \right|, \quad \forall \alpha' \le \alpha,\ \beta' \le \beta, \gamma' \le \gamma.$$

We define also

$$l^2(\mathbb{Z}; L^2(\mathbb{R})) := \left\{ u \in \tilde{\mathscr{S}}'(\Gamma) \text{ s.t. } \sum_{k \in \mathbb{Z}} |u(k, \cdot)|^2_{L^2(\mathbb{R})} < \infty \right\}.$$

Proposition 4.1. *$\tilde{\mathscr{S}}(\Gamma)$ is dense in $l^2(\mathbb{Z}; L^2(\mathbb{R}))$.*

Proof. Since $u \in l^2(\mathbb{Z}; L^2(\mathbb{R}))$, then

$$\forall \epsilon > 0 \quad \exists k_0 > 0 \text{ such that } \sum_{|k| > k_0} \| u(k, \cdot) \|^2_{L^2(\mathbb{R})} < \frac{\epsilon}{2}.$$

We set

$$u^{\delta} := \begin{cases} u^{\delta}(k, \cdot) & \text{if } |k| < k_0, \\ 0 & \text{otherwise ,} \end{cases}$$

where $u^{\delta}(k, \cdot)$ is a standard smooth approximation of $u(k, \cdot)$ in $\mathscr{D}(\mathbb{R})$. Then, we choose $\delta(k_0)$ such that

$$\forall k \text{ such that } |k| \le k_0 \quad \left\| u(k, \cdot) - u^{\delta}(k, \cdot) \right\|^2_{L^2(\mathbb{R})} < \frac{\epsilon}{2 k_0},$$

which finally gives

$$\left\| u - u^{\delta} \right\|^2_{l^2(\mathbb{Z}; L^2(\mathbb{R}))} \le \sum_{|k| \le k_0} \left\| u(k, \cdot) - u^{\delta}(k, \cdot) \right\|^2_{L^2(\mathbb{R})} + \sum_{|k| > k_0} \| u(k, \cdot) \|^2_{L^2(\mathbb{R})} < \epsilon$$

which proves the claim, since u^{δ} trivially belongs to $\tilde{\mathscr{S}}(\Gamma)$. □

Definition 4.1. *We define the MFT operator $\mathscr{F} : \mathscr{S}_{\#}(Z) \to \tilde{\mathscr{S}}(\Gamma)$ as*

$$\mathscr{F}(\varphi)(k, l) = \int_Z \varphi(\boldsymbol{y}) e^{-i(\tilde{k} y_1 + \tilde{l} y_2)} d\boldsymbol{y}, \quad \forall (k, l) \in \Gamma,$$

where $\tilde{k} = 2\pi k$, and the same holds for $\tilde{l}$.

We list below some basic tools concerning the MFT needed in the remaining of the paper. They can be proved following classical arguments ([24], [25], [26]).

Proposition 4.2. *[(i)]*

1. *The operator* $\mathscr{F}:\mathscr{S}_{\#}(Z)\to\tilde{\mathscr{S}}(\Gamma)$ *is an isomorphism and the inverse operator is explicit:*

$$\mathscr{F}^{-1}(\tilde{\varphi})(\boldsymbol{y})=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\tilde{\varphi}(k,l)e^{i(\tilde{k}y_1+\tilde{l}y_2)}dl,\quad\forall\tilde{\varphi}\in\tilde{\mathscr{S}}(\Gamma),\quad\forall\boldsymbol{y}\in Z.$$

2. *One has in a classical way, for any* $(f,g)\in\mathscr{S}_{\#}(Z)\times\mathscr{S}_{\#}(Z)$, *that*

$$\int_Z f\cdot\overline{g}\,d\boldsymbol{y}=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\mathscr{F}(f)(k,l)\cdot\overline{\mathscr{F}(g)}(k,l)dl,\ \text{and}\ \mathscr{F}(f*g)=\mathscr{F}(f)\mathscr{F}(g),\quad\forall(k,l)\in\Gamma.$$

4.2 Tempered distributions and MFT

In a quite natural manner we extend the concepts introduced above to tempered distributions.

Definition 4.2. *[(i)]*

1. *A linear form* $\tilde{T}$ *acting on* $\tilde{\mathscr{S}}(\Gamma)$ *is such that* $\exists(\alpha,\beta,\gamma)\in\mathbb{N}^3$

$$\left|\tilde{T}(\tilde{\varphi})\right|\le C\left|\tilde{\varphi}\right|_{\alpha,\beta,\gamma},\quad\forall\tilde{\varphi}\in\tilde{\mathscr{S}}(\Gamma).$$

2. *The MFT applied to a tempered distribution* $T\in\mathscr{S}'_{\#}(Z)$ *is defined as*

$$<\mathscr{F}(T),\tilde{\varphi}>_{\tilde{\mathscr{S}}',\tilde{\mathscr{S}}}:=\mathscr{F}(T)(\tilde{\varphi}):=T(\mathscr{F}^T(\tilde{\varphi}))=<T,\mathscr{F}^T(\tilde{\varphi})>_{\mathscr{S}'_{\#}\times\mathscr{S}_{\#}},\quad\forall\tilde{\varphi}\in\tilde{\mathscr{S}}(\Gamma),$$

where

$$\mathscr{F}^T(\tilde{\varphi}):=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\tilde{\varphi}(k,l)e^{-i(\tilde{k}y_1+\tilde{l}y_2)}dl,\quad\forall\tilde{\varphi}\in\tilde{\mathscr{S}}(\Gamma),\quad\forall\boldsymbol{y}\in Z.$$

In the same way, we define the reciprocal operator denoted $\mathscr{F}^{-1}$ *which associates to each distribution* $\tilde{T}\in\tilde{\mathscr{S}}'(\Gamma)$ *a distribution* $\mathscr{F}^{-1}(\tilde{T})$ *s.t.*

$$<\mathscr{F}^{-1}(\tilde{T}),\varphi>_{\mathscr{S}'\times\mathscr{S}}=<\tilde{T},\check{\mathscr{F}}(\varphi)>_{\tilde{\mathscr{S}}'\times\tilde{\mathscr{S}}}$$

where $\check{\tilde{\varphi}}(k,l)=\tilde{\varphi}(-k,-l)$.

We sum up properties extending classical results of Fourier analysis to our MFT.

Proposition 4.3. *[(i)]*

1. *If* $T\in\mathscr{S}'_{\#}(Z)$ *then* $\mathscr{F}(T)\in\tilde{\mathscr{S}}'(\Gamma)$. *Similarly, if* $\tilde{T}\in\tilde{\mathscr{S}}'(\Gamma)$ *then* $\mathscr{F}^{-1}(\tilde{T})\in\mathscr{S}'_{\#}(Z)$.

2. *The Dirac measure belongs to* $\mathscr{S}'_{\#}(Z)$ *and :*

$$\mathscr{F}(\delta_0)=1,\quad\forall k\in\mathbb{Z},\quad a.e.\ \ell\in\mathbb{R}.$$

3. *the MFT acts on the derivatives in a polynomial fashion:*

$$\forall T \in \mathscr{S}'_{\#}(Z), \quad \mathscr{F}(\partial^{\alpha} T) = i^{|\alpha|}(\tilde{k}^{\alpha} + \tilde{l}^{\alpha})\mathscr{F}(T),$$

for any multi-index $\alpha \in \mathbb{N}^2$.

4. *Plancherel's Theorem : if* $f \in L^2(Z)$ *then the Fourier transform* $\mathscr{F}(T_f)$ *is defined by a function* $\mathscr{F}(f) \in l^2(\mathbb{Z}, L^2(\mathbb{R}))$ *i.e.*

$$\mathscr{F}(T_f) = T_{\mathscr{F}(f)} \quad \forall f \in L^2(Z) \text{ and } \|\mathscr{F}(f)\|_{l^2(\mathbb{Z};L^2(\mathbb{R}))} = \|f\|_{L^2(Z)}.$$

5 The Laplace equation in a periodic infinite strip

In this section we study the problem

$$\begin{cases} -\Delta u = f \text{ in } Z, \\ \ \ u \text{ is 1-periodic in the } y_1 \text{ direction} \end{cases} \tag{5.20}$$

in the variational context. Firstly, we characterize of the kernel of the Laplace operator.

Proposition 5.1. *Let* $m \geq 0$ *be an integer,* α *be a real number and* $j = \min\{q(m,\alpha), 1\}$ *where* $q(m,\alpha)$ *is defined by Proposition 2.2. A function* $u \in H^m_{\alpha,\#}(Z)$ *satisfies* $\Delta u = 0$ *if and only if* $u \in \mathbb{P}'^{\Delta}_j$.

Proof. Since $j \leq 1$, it is clear that if $u \in \mathbb{P}'^{\Delta}_j$, then $\Delta u = 0$. Conversely, let $u \in \mathscr{S}'_{\#}(Z)$ satisfy $\Delta u = 0$. We apply the MFT : in the sense defined in Definition 4.2, one has

$$(\tilde{k}^2 + \tilde{l}^2)\mathscr{F}(u) = 0, \quad \forall k \in \mathbb{Z}, \text{ a.e. } l \in \mathbb{R},$$

which implies that

$$\mathscr{F}(u) = \begin{cases} \sum_{j=0}^{p} \delta^{j}_{y_2=0}(l) & \text{if } k \equiv 0, \\ 0 & \text{otherwise,} \end{cases}$$

p being a non negative integer. A simple computation shows that $\mathscr{F}^{-1}(\mathscr{F}(u)) = \sum_{j=0}^{p}(iy_2)^j$, indeed

$$\begin{aligned} &\langle \mathscr{F}^{-1}(\mathscr{F}(u)), \varphi\rangle_{\mathscr{S}'_{\#},\mathscr{S}_{\#}} = \langle \mathscr{F}(u), \breve{\mathscr{F}}(\varphi)\rangle_{\breve{\mathscr{S}}'(\Gamma),\breve{\mathscr{S}}(\Gamma)} = \langle \sum_{j=0}^{p} \delta_0^j, \breve{\mathscr{F}}(\varphi)(0,\cdot)\rangle_{\mathscr{S}'(\mathbb{R})\times\mathscr{S}(\mathbb{R})} \\ &= \sum_{j=0}^{p} (-1)^j \langle \delta_0, \partial_j \hat{\varphi}\rangle_{\mathscr{S}'(\mathbb{R})\times\mathscr{S}(\mathbb{R})} = \sum_{j=0}^{p} (-1)^j \langle \delta_0, \widehat{((iy_2)^j \varphi)}\rangle_{\mathscr{S}'(\mathbb{R})\times\mathscr{S}(\mathbb{R})} \\ &= \sum_{j=0}^{p} (-1)^j \int_{\mathbb{R}} (iy_2)^j \varphi(\mathbf{y})\, d\mathbf{y} \quad = \langle \sum_{j=0}^{p} (-iy_2)^j, \varphi\rangle_{\mathscr{S}'_{\#}\times\mathscr{S}_{\#}}, \end{aligned}$$

where $\hat{\varphi}$ denotes the usual 1-dimensional Fourier transform of φ on $\mathbb{R}$. □

Theorem 5.1. *Assume α be a real number such that $-1/2 \le \alpha \le 1/2$. If $f \in H^{-1}_{\alpha,\#}(Z)\perp\mathbb{R}$ then problem (1.1) has a unique solution $u \in H^1_{\alpha,\#}(Z)/\mathbb{R}$,* i.e. *the Laplace operator defined by*

$$\Delta : H^1_{\alpha,\#}(Z)/\mathbb{R} \mapsto H^{-1}_{\alpha,\#}(Z)\perp\mathbb{R}, \tag{5.21}$$

is an isomorphism.

Proof. Let us first check that if u belongs to $H^1_{\alpha,\#}(Z)$ for $-1/2 \le \alpha \le 1/2$, then Δu belongs to $H^{-1}_{\alpha,\#}(Z)$. Indeed, for any $\varphi \in \mathcal{D}_\#(Z)$, we can write

$$|\langle \Delta u, \varphi\rangle_{\mathcal{D}'_\#(Z),\mathcal{D}_\#(Z)}| = |\langle \nabla u, \nabla\varphi\rangle_{L^2_\alpha(Z),L^2_{-\alpha}(Z)}| \le \|\nabla u\|_{L^2_\alpha(Z)}\|\nabla\varphi\|_{L^2_{-\alpha}(Z)} = \|\nabla u\|_{L^2_\alpha(Z)}|\varphi|_{H^1_{-\alpha,\#}(Z)}.$$

Since $-1/2 \le \alpha \le 1/2$, then thanks to Theorem 3.1, the semi-norm $|.|_{H^1_{-\alpha,\#}(Z)}$ is a norm on $H^1_{-\alpha,\#}(Z)/\mathbb{R}$. Therefore, we deduce that $\Delta u \in H^{-1}_{\alpha,\#}(Z)\perp\mathbb{R}$.
Observe now that for any function $u \in H^1_{\alpha,\#}(Z)$, there exists a constant k s.t.

$$\|u-k\|_{L^2_{\alpha-1}(Z)} = \inf_{c\in\mathbb{R}} \|u-c\|_{L^2_{\alpha-1}(Z)}.$$

Since the functional is convex wrt c one derives easily that $k(u)$ necessarily satisfies :

$$k(u) = \frac{\displaystyle\int_Z u\rho^{(2\alpha-2)}\,d\boldsymbol{y}}{\displaystyle\int_Z \rho^{(2\alpha-2)}\,d\boldsymbol{y}}.$$

A simple use of Jensen inequality proves that

$$(k(u))^2 \le C(\alpha)\|u\|^2_{H^0_{\alpha-1,\#}(Z)}.$$

Next, we first prove the statement for $-1/2 \le \alpha \le 0$. To do so, we prove the inf-sup condition. Set $\omega := \rho^\alpha$ and define

$$v(u) := \omega^2(u-k(u)).$$

Let us prove that $v(u) \in H^1_{-\alpha,\#}(Z)$ for any $-1/2 \le \alpha \le 0$. It is clear that

$$\|v(u)\|_{L^2_{-\alpha-1}(Z)} = \|u-k(u)\|_{L^2_{\alpha-1}(Z)}.$$

Consequently, for the particular case $\alpha = -1/2$ we also have

$$\|\ln(1+\rho^2)^{-1}v(u)\|_{L^2_{-1/2}(Z)} \le C\|u-k(u)\|_{L^2_{-3/2}(Z)}. \tag{5.22}$$

Note that inequality (5.22) is needed, since logarithmic weight appears in the space $H^1_{1/2,\#}(Z)$ (see (2.8)). Furthermore, direct computations show that $\nabla v(u) = 2\omega\nabla\omega(u-k(u)) + \omega^2\nabla u$ and $\partial_{y_2}\omega = \alpha y_2\rho^{\alpha-2}$ which implies that $|\nabla\omega| \le C\rho^{\alpha-1}$. It follows that

$$\|\nabla v(u)\|_{L^2_{-\alpha}(Z)} \le C\|u-k(u)\|_{H^1_{\alpha,\#}(Z)}.$$

This shows that for any $-1/2 \le \alpha \le 0$, $v(u) \in H^1_{-\alpha,\#}(Z)$ and

$$\| v(u) \|_{H^1_{-\alpha,\#}(Z)} \le C \| u - k(u) \|_{H^1_{\alpha,\#}(Z)}. \tag{5.23}$$

Now, one has :

$$a(u,v) = \int_Z \nabla u \cdot \nabla v \, d\boldsymbol{y} = \int_Z |\nabla u|^2 \omega^2 + \nabla u \cdot (\nabla \omega^2)(u - k(u)) \, d\boldsymbol{y}.$$

One focuses on the last term of the rhs, uses the divergence formula and writes

$$\int_Z \nabla u \cdot (\nabla \omega^2)(u - k(u)) d\boldsymbol{y} = -\frac{1}{2} \int_Z \Delta(\omega^2)(u - k(u))^2 d\boldsymbol{y} = -\int_Z \left\{ (\Delta\omega)\omega + |\nabla \omega|^2 \right\} (u - k(u))^2 d\boldsymbol{y}$$

Using again that $\partial_{y_2} \omega = \alpha y_2 \rho^{\alpha-2}$ one has explicitly that

$$(\Delta\omega)\omega + |\nabla\omega|^2 = \left(2(\alpha^2 - 1) \frac{y^2}{\rho^2} + \alpha \right) \rho^{(2\alpha - 2)}.$$

Since $-1/2 \le \alpha \le 0$, we clearly have

$$(\Delta\omega)\omega + |\nabla\omega|^2 \le 0.$$

This allows to write:

$$a(u, v(u)) \ge \int_Z |\nabla u|^2 \omega^2 \, d\boldsymbol{y}.$$

Thanks to Theorem 3.1 and inequality (5.23), we can write

$$a(u, v(u)) \ge \int_Z |\nabla u|^2 \omega^2 d\boldsymbol{y} \ge C \| u \|_{H^1_{\alpha,\#}(Z)/\mathbb{R}} \| v(u) \|_{H^1_{-\alpha,\#}(Z)}.$$

The above inequality allows to obtain the inf-sup condition which proves that the Laplace operator defined by

$$\Delta : H^1_{\alpha,\#}/\mathbb{R} \mapsto H^{-1}_{\alpha,\#}(Z) \perp \mathbb{R} \tag{5.24}$$

is an isomorphism provided that $-1/2 \le \alpha \le 0$. By duality and transposition, the Laplace operator defined by (5.24) is also an isomorphism for $0 \le \alpha \le 1/2$. This ends the proof. □

It remains to extend the isomorphism result (5.21) to any $\alpha \in \mathbb{R}$ and to any $m \in \mathbb{Z}$. For this sake, we state some regularity results for the Laplace operator in Z.

Theorem 5.2. *Let α be a real number such that $-1/2 < \alpha < 1/2$ and let ℓ be an integer. Then the Laplace operator defined by*

$$\Delta : H^{1+\ell}_{\alpha+\ell,\#}(Z)/\mathbb{R} \mapsto H^{-1+\ell}_{\alpha+\ell,\#}(Z) \perp \mathbb{R} \tag{5.25}$$

is an isomorphism.

Proof. Owing to Theorem 5.1, the claim is true for $\ell = 0$. Assume that it is true for $\ell = k$ and let us prove that it is still true for $\ell = k+1$. The Laplace operator defined by (5.25) is clearly linear and continuous. It is also injective : if $u \in H^{k+2}_{\alpha+k+1,\#}(Z)$ and $\Delta u = 0$ then u is constant. To prove that it is onto, let f be given in

$H^{k}_{\alpha+k+1,\#}(Z)\perp\mathbb{R}$. According to (2.9), f belongs to $H^{-1+k}_{\alpha+k,\#}(Z)\perp\mathbb{R}$. Then the induction assumption implies that there exists $u \in H^{1+k}_{\alpha+k,\#}(Z)$ such that $\Delta u = f$. Next, we have

$$\Delta(\rho\partial_i u) = \rho\partial_i f + \partial_i u\Delta\rho + 2\nabla\rho\cdot\nabla(\partial_i u). \tag{5.26}$$

Using (2.6), (2.9), (2.11) and (2.12), all the terms of the right-hand side belong to $H^{-1+k}_{\alpha+k,\#}(Z)$. This implies that $\Delta(\rho\partial_i u)$ belongs to $H^{-1+k}_{\alpha+k,\#}(Z)$. Therefore, $\Delta(\rho\partial_i u)$ also belongs to $H^{-2+k}_{\alpha+k-1,\#}(Z)$. Moreover, u belonging to $H^{1+k}_{\alpha+k,\#}(Z)$ implies that $\rho\partial_i u$ belongs $H^{k}_{\alpha+k-1,\#}(Z)$ and for any $\varphi \in H^{2-k}_{-\alpha-k+1,\#}(Z)$, we have

$$\langle\Delta(\rho\partial_i u),\varphi\rangle_{H^{-2+k}_{\alpha+k-1,\#}(Z)\times H^{2-k}_{-\alpha-k+1,\#}(Z)} = \langle\rho\partial_i u,\Delta\varphi\rangle_{H^{k}_{\alpha+k-1,\#}(Z)\times H^{-k}_{-\alpha-k+1,\#}(Z)}.$$

Since $\mathbb{R}\subset H^{2-k}_{-\alpha-k+1,\#}(Z)$, we can take $\varphi\in\mathbb{R}$ which implies that $\Delta\varphi = 0$. It follows that $\Delta(\rho\partial_i u)$ belongs to $H^{-1+k}_{\alpha+k,\#}(Z)\perp\mathbb{R}$. Thanks to the induction assumption, there exists v in $H^{1+k}_{\alpha+k,\#}(Z)$ such that

$$\Delta v = \Delta(\rho\partial_i u).$$

Hence, the difference $v-\rho\partial_i u$ is a constant. Since the constants are in $H^{1+k}_{\alpha+k,\#}(Z)$, we deduce that $\rho\partial_i u$ belongs to $H^{1+k}_{\alpha+k,\#}(Z)$ which implies that u belongs to $H^{2+k}_{\alpha+k+1,\#}(Z)$.

□

Remark 5.1. Let us point out that the above theorem excludes the values $\alpha\in\{-\frac{1}{2},\frac{1}{2}\}$: due to logarithmic weights, the space $H^{1}_{1/2,\#}(Z)$ is not included in $L^{2}_{-1/2}(Z)$. By duality, this also implies that the space $L^{2}_{1/2}(Z)$ is not included in $H^{-1}_{-1/2,\#}(Z)$. Therefore, in (5.26) for $\alpha = 1/2$, the term $\partial_i u\Delta\rho$ does not belong to $H^{-1}_{1/2,\#}(Z)$. Thus, in this paper, the extension of (5.21) to any $\alpha\in\mathbb{R}$ will exclude some values of α that are called critical values and belong to the set $\mathbb{Z}+\frac{1}{2}$. Furthermore, tough (5.21) is valid for the critical values $\alpha=\pm\frac{1}{2}$, we shall exclude these values here for sake of clarity. We deal with the critical cases in a forthcoming paper.

Remark 5.2. Applying the theorem above to the particular case when $\ell = 1$, we derive a weighted version of Calderón-Zygmund inequalities [27]. More precisely, let α be a real number such that $1/2<\alpha<3/2$, then there exists a constant $C>0$, such that for any $u\in\mathcal{D}_{\#}(Z)$:

$$\left\|\partial_i\partial_j u\right\|_{L^2_\alpha(Z)} \le C\|\Delta u\|_{L^2_\alpha(Z)}. \tag{5.27}$$

Using (5.27) and thanks to the Closed Range Theorem of Banach, we prove that :

Theorem 5.3. *Let α be a real number satisfying $1/2<\alpha<3/2$. For $m\in\mathbb{N}$, $m\ge 3$, the following operator*

$$\Delta : H^{m}_{\alpha,\#}(Z)/\mathbb{P}'_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z)/\mathbb{P}'_{m-4} \tag{5.28}$$

is an isomorphism.

The next claim is then a straightforward consequence of the latter result.

Theorem 5.4. *Let α be a real number satisfying $1/2 < \alpha < 3/2$. For $m \in \mathbb{N}$, $m \geq 3$, the following operator*

$$\Delta : H^m_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z) \tag{5.29}$$

is an isomorphism.

The next theorem extends Theorem 5.1.

Theorem 5.5. *Let α be a real number satisfying $1/2 < \alpha < 3/2$ and let $\ell \geq 1$ be a given integer. Then the Laplace operators defined by*

$$\Delta : H^1_{-\alpha+\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{1-\ell} \mapsto H^{-1}_{-\alpha+\ell,\#}(Z)\perp\mathbb{P}'^{\Delta}_{-1+\ell} \tag{5.30}$$

and

$$\Delta : H^1_{\alpha-\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{-1+\ell} \mapsto H^{-1}_{\alpha-\ell,\#}(Z)\perp\mathbb{P}'^{\Delta}_{1-\ell} \tag{5.31}$$

are isomorphisms.

Proof. Observe first that when $\ell = 1$, the result is proved in Theorem 5.1. Observe next that if $m \geq 2$ is an integer, the mapping

$$\Delta : H^m_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{m-2} \mapsto H^{m-2}_{\alpha,\#}(Z)\perp\mathbb{P}'^{\Delta}_{-m+2}$$

is onto. Indeed, if $m = 2$, this isomorphism is exactly defined by (5.25) with $\ell = 1$. If $m \geq 3$, it is defined by (5.29). Now through duality and transposition, the mapping

$$\Delta : H^{-m+2}_{-\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{-m+2} \mapsto H^{-m}_{-\alpha,\#}(Z)\perp\mathbb{P}'^{\Delta}_{m-2}$$

is onto. Next, using the same arguments as in the proof of Theorem 5.2, we are able to show that for any integer $\ell \geq 1$, the operator

$$\Delta : H^{-m+2+\ell}_{-\alpha+\ell,\#}(Z)/\mathbb{P}'^{\Delta}_{-m+2} \mapsto H^{-m+\ell}_{-\alpha+\ell,\#}(Z)\perp\mathbb{P}'^{\Delta}_{m-2}$$

is an isomorphism. Choosing $m = \ell + 1$, the operator defined by (5.30) is an isomorphism. By duality and transposition, the mapping defined by (5.31) is onto as well.

□

Remark 5.3. Summarizing theorems 5.1 and 5.5, we deduce that, for any $\alpha \in \mathbb{R}$ such that $\alpha \notin \mathbb{Z} + \frac{1}{2}$, the mapping

$$\Delta : H^1_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[1/2-\alpha]} \mapsto H^{-1}_{\alpha,\#}(Z)\perp\mathbb{P}'^{\Delta}_{[1/2+\alpha]} \tag{5.32}$$

is an isomorphism . As a consequence, for any $m \in \mathbb{Z}$, for any $\alpha \in \mathbb{R}$ such that $\alpha \notin \mathbb{Z} + \frac{1}{2}$, the mapping

$$\Delta : H^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]} \mapsto H^m_{\alpha,\#}(Z)\perp\mathbb{P}'^{\Delta}_{[-m-1/2+\alpha]} \tag{5.33}$$

is an isomorphism. This result is stated in Theorem 1.1.

We end this section with a proposition that gives a property of all distribution $u \in \mathscr{D}'_\#(Z)$ satisfying $\nabla u \in L^2_{-\frac{1}{2}}(Z)$. The proof of the proposition is given in E.

Proposition 5.2. *Let $u \in \mathscr{D}'_\#(Z)$ such that $\nabla u \in L^2_{-\frac{1}{2}}(Z)$. Then $u \in H^1_{-\frac{1}{2},\#}(Z)$ and there exists a constant $C > 0$ such that*

$$\|u\|_{H^1_{-\frac{1}{2},\#}(Z)/\mathbb{R}} \le C\|\nabla u\|_{L^2_{-\frac{1}{2}}(Z)}.$$

6 The Green function and convolution in weighted spaces

The purpose of this section is threefold:

(i) we look for the fundamental solution for the Laplace equation in Z

(ii) we estimate the convolution with this solution in weighted Sobolev spaces

(iii) we identify the solution of (1.1) found in the previous section with the convolution between the fundamental solution ans the datum f. This amount to prove Theorem 1.2.

To achieve the first goal, we use the MFT defined in Section 4.

6.1 The Green function for periodic strips

We aim at solving the fundamental equation:

$$-\Delta G = \delta_0 \quad \text{in } Z, \tag{6.34}$$

stated here in the sense of tempered distributions. Then G should satisfy, in $\tilde{\mathscr{S}}(\Gamma)'$:

$$(\tilde{k}^2 + \tilde{l}^2)\mathscr{F}(G) = 1.$$

The proposition below gives the expression of the Green function.

Proposition 6.1. *The Green function G solving (6.34) is a tempered distribution and it reads*

$$G(\boldsymbol{y}) = \frac{1}{2}\left\{\sum_{k\in\mathbb{Z}^*} \frac{e^{-|\tilde{k}||y_2|+i\tilde{k}y_1}}{|\tilde{k}|} - |y_2|\right\} = G_1(\boldsymbol{y}) + G_2(\boldsymbol{y}), \quad \forall \boldsymbol{y} \in Z,$$

where

$$G_1(\boldsymbol{y}) := \frac{1}{2}\sum_{k\in\mathbb{Z}^*} \frac{e^{-|\tilde{k}||y_2|+i\tilde{k}y_1}}{|\tilde{k}|} \quad \textit{and} \quad G_2(\boldsymbol{y}) := -\frac{1}{2}|y_2|.$$

Moreover, one has $G_1 \in L^1(Z) \cap L^2(Z)$ and thus $G \in L^2_{\text{loc}}(Z)$. The Green function can be written in a more compact expression :

$$G(\boldsymbol{y}) = -\frac{1}{4\pi}\ln\left(2\left(\cosh(2\pi y_2) - \cos(2\pi y_1)\right)\right).$$

Notice that the Green function is odd with respect to $\boldsymbol{y}$.

The proof of this proposition is given in B. One can easily obtain the asymptotic behavior of the Green function. As $|y_2|$ tends to infinity, we have

$$|G(\boldsymbol{y})| \leq C|y_2|, \quad |\nabla G(\boldsymbol{y})| \leq C \text{ and } |\partial^2 G(\boldsymbol{y})| \leq Ce^{-2\pi|y_2|} \leq C\rho(y_2)^{-\sigma}, \ \forall \sigma \in \mathbb{R}. \tag{6.35}$$

Note that the estimate on the derivatives of second order on G is proved in F (Lemma 1). Moreover, as $|y_2|$ tends to infinity, we also have

$$|G_1(\boldsymbol{y})| \leq C\rho(y_2)^{-\sigma}, \ \forall \sigma > 0, \quad \forall y_1 \in (0,1). \tag{6.36}$$

Before proving weighted estimates on the Green function, we shall define the convolution if $f \in \mathscr{D}_\#(Z)$, i.e.

$$G * f = \int_Z G(\boldsymbol{x}-\boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y} = \int_Z G(\boldsymbol{y})\, f(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y}.$$

Thus the convolution $G * f$ belongs to $\mathscr{C}^\infty_\#(Z)$. Moreover, thanks to (6.34), we have $\Delta(G * f) = f$ in Z.

6.2 First properties of the convolution with the fundamental solution

Definition 6.1. *For any function f in $L^2(Z)$ the horizontal average $\overline{f}$ reads :*

$$\overline{f}(y_2) := \int_0^1 f(y_1, y_2)\, dy_1.$$

We continue with a property of the convolution between the Green function and data that have zero horizontal average.

Lemma 6.1. *For any $h \in \mathscr{D}_\#(Z)$ such that $\overline{h} = 0$ one has :*

$$\partial^{\boldsymbol{\alpha}} G_2 * h = 0, \text{ a.e. in } Z, \quad \boldsymbol{\alpha} \in \mathbb{N}^2 \text{ s.t. } |\boldsymbol{\alpha}| \in \{0,1,2\}.$$

Proof. G_2 is a tempered distribution, the convolution with $h \in \mathscr{D}_\#(Z)$ makes sense :

$$\begin{aligned} < G_2 * h, \varphi >_{\mathscr{D}'_\#(Z)\times\mathscr{D}_\#(Z)} &= < G_2, (h \check{*} \varphi) >_{\mathscr{D}'_\#(Z)\times\mathscr{D}_\#(Z)} \\ &= \frac{1}{2}\int_Z |x_2| \int_Z h(\boldsymbol{x}+\boldsymbol{y})\varphi(\boldsymbol{y})\, d\boldsymbol{y} d\boldsymbol{x} = \frac{1}{2}\int_{\mathbb{R}} |x_2| \int_Z \overline{h}(x_2+y_2)\, \varphi(\boldsymbol{y})\, d\boldsymbol{y} dx_2 = 0. \end{aligned}$$

The same proof holds for derivatives as well. As this is true for every $\varphi \in \mathscr{S}_\#(Z)$, the result is proved. □

By the same way one has the complementary counterpart :

Lemma 6.2. *For any $h \in \mathscr{D}_\#(Z)$ such that h only depends on y_2 one has :*

$$\partial^{\boldsymbol{\alpha}} G_1 * h = 0, \text{ a.e. in } Z, \quad \boldsymbol{\alpha} \in \mathbb{N}^2 \text{ s.t. } \boldsymbol{\alpha} \in \{0,1,2\}.$$

Lemma 6.3. *We define the horizontal Fourier transform, given $f \in L^2(Z)$*

$$\mathscr{F}_k(f)(k, y_2) := \int_0^1 f(y_1, y_2) e^{-i\tilde{k}y_1}\, dy_1, \quad \forall k \in \mathbb{Z}, \quad a.e\ y_2 \in \mathbb{R}.$$

If we define the convolution operators : for $g \in L^1(Z) \cap L^2(Z)$ and $f \in L^2(Z)$

$$g *_{\boldsymbol{y}} f := \int_Z g(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y}) d\boldsymbol{y}, \quad (g *_{y_2} f)(x_1, x_2, z_1) := \int_{\mathbb{R}} g(x_1, x_2 - y_2) f(z_1, y_2) dy_2,$$

then one has

$$\mathscr{F}_k(g *_{\boldsymbol{y}} f)(k, x_2) = (\mathscr{F}_k g *_{y_2} \mathscr{F}_k f)(k, x_2), \quad \forall k \in \mathbb{Z}, \quad a.e.\ x_2 \in \mathbb{R}.$$

Proof. If $f \in L^2(Z)$ and $g \in (L^1 \cap L^2)(Z)$ then the convolution with respect to both variables is well defined. Setting

$$(f *_{y_1} g)(x_1, x_2, z_2) := \int_{\mathbb{R}} f(x_1 - y_1, x_2) g(y_1, z_2) dy_1,$$

almost everywhere in y_2 and for every $k \in \mathbb{Z}$ one has

$$\mathscr{F}_k \left(f *_{y_1} g(x_1, x_2 - y_2, y_2) \right) = (\mathscr{F}_k f)(k, x_2 - y_2)(\mathscr{F}_k g)(k, y_2).$$

Integrals in the vertical direction commute with the horizontal Fourier transform due to Fubini's theorem. So one shall write :

$$\begin{aligned} \int_{\mathbb{R}} \mathscr{F}_k \left(f *_{y_1} g \right)(k, x_2 - y_2, y_2) dy_2 &= \mathscr{F}_k \left(\int_{\mathbb{R}} (f *_{y_1} g)(k, x_2 - y_2, y_2) dy_2 \right) \\ &= \mathscr{F}_k (f *_{\boldsymbol{y}} g)(k, x_2), \end{aligned}$$

which ends the proof. □

Lemma 6.4. *For $f \in L^2_\beta(Z)$, one has*

$$\|f\|^2_{L^2_\beta(Z)} = \sum_{k \in \mathbb{Z}} \|\mathscr{F}_k(f)\|^2_{L^2_\beta(\mathbb{R})}, \quad \forall \beta \in \mathbb{R}.$$

The proof the above lemma is given in C.

Lemma 6.5. *Let $a \in \mathbb{R}$ s.t. $|a| \geq 1$ then for any real b, one has*

$$I(x) := \int_{\mathbb{R}} e^{-|a||x-y|} \rho(y)^{-b} dy \leq \frac{C \rho(x)^{-b}}{|a|}, \quad x \in \mathbb{R}.$$

6.3 Non-homogeneous estimates

Proposition 6.2. *Let us set β a real number. Then for every $f \in \mathscr{D}_\#(Z)$*

$$\left\| \partial^\alpha G * (f - \overline{f}) \right\|_{L^2_{\beta'}(Z)} \leq C \left\| f - \overline{f} \right\|_{L^2_\beta(Z)}, \quad \forall \beta' < \beta - \frac{1}{2}, \quad |\alpha| \in \{0, 1\},$$

where the constant C is independent of the data.

Proof. Set again $h := f - \overline{f}$. By Lemma 6.4 :

$$\begin{aligned} \|\partial^\alpha G * h\|^2_{L^2_{\beta'}(Z)} = \|\partial^\alpha G_1 * h\|^2_{L^2_{\beta'}(Z)} &= \sum_{k \in \mathbb{Z}^*} \left\| \mathscr{F}_k(\partial^\alpha_{\boldsymbol{y}} G_1 *_{\boldsymbol{y}} h) \right\|^2_{L^2_{\beta'}(\mathbb{R})} \\ &= \sum_{k \in \mathbb{Z}^*} \left\| (i\tilde{k})^{\alpha_1} \mathscr{F}_k(\partial^{\alpha_2}_{y_2} G_1) *_{y_2} \mathscr{F}_k(h) \right\|^2_{L^2_{\beta'}(\mathbb{R})}. \end{aligned}$$

For all $k \in \mathbb{Z}^*$, thanks to Lemma 6.5, a.e. $\boldsymbol{x}$ in Z,

$$\left|(i\tilde{k})^{\alpha_1}\mathscr{F}_k(\partial_{y_2}^{\alpha_2}G_1) *_{y_2} \mathscr{F}_k(h)\right| \leq \int_{\mathbb{R}} \frac{e^{-|\tilde{k}||y_2-x_2|}}{|\tilde{k}|^{1-|\alpha|}} \mathscr{F}_k(h)(k,y_2)\,dy_2$$
$$\leq C\|\mathscr{F}_k(h)(k,\cdot)\|_{L^2_\beta(\mathbb{R})} \frac{\rho(x_2)^{-\beta}}{|\tilde{k}|^{\frac{3}{2}-|\alpha|}} \leq C\|\mathscr{F}_k(h)(k,\cdot)\|_{L^2_\beta(\mathbb{R})}\rho(x_2)^{-\beta},$$

for α multi-index s.t. $|\alpha| \in \{0,1\}$. Then

$$\left\|\partial^\alpha G * h\right\|^2_{L^2_{\beta'}(Z)} \leq \sum_{k\in\mathbb{Z}^*} \|\mathscr{F}_k(h)(k,\cdot)\|^2_{L^2_\beta(\mathbb{R})} \int_{\mathbb{R}} \rho(x_2)^{2\beta'-2\beta}\,dx_2 \leq C\left\|f\right\|_{L^2_\beta(Z)},$$

if $\beta' < \beta - 1/2$. □

We first have the following result.

Theorem 6.1. *Let α be a real number such that $\alpha > 3/2$ and $f \in \mathscr{D}_\#(Z)$, then for any $\epsilon > 0$ one has*

$$\left\|G_2 * \overline{f} - H(\cdot,\overline{f})\right\|_{L^2_{\alpha-2-\epsilon}(Z)} \leq C\left\|f\right\|_{L^2_\alpha(Z)}$$

where

$$H(x_2,f) = \frac{1}{2}\left(|x_2|\int_{\mathbb{R}} \overline{f}(y_2)\,dy_2 - \operatorname{sgn}(x_2)\int_{\mathbb{R}} y_2\overline{f}(y_2)\,dy_2\right).$$

Furthermore, if f is orthogonal to polynomials of $\mathbb{P}'_1$, namely $f \in \mathscr{D}_\#(Z)\perp\mathbb{P}'_1$, then $H(.,f) = 0$ and the above estimate is reduced to

$$\|G_2 * \overline{f}\|_{L^2_{\alpha-2-\epsilon}(Z)} \leq C\|f\|_{L^2_\alpha(Z)}.$$

Proof. As the argumentation is purely one-dimensional, we denote by f the average of f in the y_1 direction omitting the overline symbol. Note that we can write the convolution $|x_2| * f$ explicitly:

$$(|x_2| * f) = \int_{-\infty}^{x_2} (x_2 - y_2) f(y_2)\,dy_2 - \int_{x_2}^{\infty} (x_2 - y_2) f(y_2)\,dy_2$$
$$= x_2\left(\int_{-\infty}^{x_2} f(y_2)\,dy_2 - \int_{x_2}^{\infty} f(y_2)\,dy_2\right) + \left(-\int_{-\infty}^{x_2} y_2 f(y_2)\,dy + \int_{x_2}^{\infty} y_2 f(y_2)\,dy_2\right).$$

Pointwize considerations show that

$$\frac{1}{2}|x_2| * f \sim H(x_2,f), \text{ when } |x_2| \to \infty.$$

Indeed,

$$\frac{|x_2| * f}{|x_2|} \sim \frac{x_2\left(\int_{-\infty}^{x_2} f(y_2)\,dy_2 - \int_{x_2}^{\infty} f(y_2)\,dy_2\right)}{|x_2|} \to \int_{\mathbb{R}} f(y_2)\,dy_2 \text{ Âăwhen } |x_2| \to \infty$$

and

$$\left(-\int_{-\infty}^{x_2} y_2 f(y_2)\,dy_2 + \int_{x_2}^{\infty} y_2 f(y_2)\,dy_2\right) \to -\operatorname{sgn}(x_2)\int_{\mathbb{R}} y_2 f(y_2)\,dy_2 \text{ ăwhen } |x_2| \to \infty.$$

A more specific calculation shows that actually

$$\frac{1}{2}|x_2| * f - H(x_2,f) = |x_2|\left(\int_{-\infty}^{x_2} f(y_2)\,dy_2\,\mathbb{1}_{\mathbb{R}_-}(x_2) - \int_{x_2}^{\infty} f(y_2)\,dy_2\,\mathbb{1}_{\mathbb{R}_+}(x_2)\right)$$
$$+ \operatorname{sgn}(x_2)\left(\int_{-\infty}^{x_2} y_2 f(y_2)\,dy_2\,\mathbb{1}_{\mathbb{R}_-}(x_2) + \int_{x_2}^{\infty} y_2 f(y_2)\,dy_2\,\mathbb{1}_{\mathbb{R}_+}(x_2)\right).$$

Provided that α belongs to the correct interval, an easy use of Hölder estimates shows then that:

$$\left|\frac{1}{2}|x_2| * f - H(x_2, f)\right| \leq C\rho^{\frac{3}{2}-\alpha}(x_2)\|f\|_{L^2_\alpha(Z)}, \quad \forall x_2 \in \mathbb{R}$$

which then multiplying by the appropriate weight on both sides and integrating wrt x_2 gives the desired result. □

The below lemma is used in the next theorem and also to get weighted estimates on the convolution with the derivative of the Green function.

Lemma 6.6. *Let $\alpha \in \mathbb{R}$, $f \in \mathcal{D}_\#(Z)$ and ϵ a positive real parameter.*

1. *If $\alpha > \frac{1}{2}$ and if f is orthogonal to $\mathbb{R}$, namely $f \in \mathcal{D}_\#(Z)\perp\mathbb{R}$, then*

$$\left\|\mathrm{sgn} * \overline{f}\right\|_{L^2_{\alpha-1-\epsilon}(Z)} \leq C\|f\|_{L^2_\alpha(Z)}.$$

2. *If $\alpha \leq \frac{1}{2}$, then*

$$\left\|\mathrm{sgn} * \overline{f}\right\|_{L^2_{\alpha-1-\epsilon}(Z)/\mathbb{R}} \leq C\|f\|_{L^2_\alpha(Z)}.$$

Proof. [(i)]

Suppose in a first step that f does not satisfy the polar condition $f\perp\mathbb{R}$. For $\alpha > \frac{1}{2}$, the constants belong to $L^2_{-\alpha}$, this implies in an obvious way that $\int_{\mathbb{R}} \overline{f}(y_2)dy_2$ is well defined. Then one writes :

$$(\mathrm{sgn}(x_2) * \overline{f})(\boldsymbol{x}) = \int_{y_2<x_2} \overline{f}(y_2)\,dy_2 - \int_{y_2>x_2} \overline{f}(y_2)\,dy_2.$$

For x_2 tending to infinity, $\overline{f} * \mathrm{sgn}(x_2)$ behaves as $\mathrm{sgn}(x_2)\int_{\mathbb{R}} \overline{f}$. Indeed

$$\begin{aligned}
&\left|(\mathrm{sgn}(x_2) * \overline{f})(\boldsymbol{x}) - \mathrm{sgn}(x_2)\int \overline{f}(s)ds\right| \\
&= 2\left|\left(\int_{-\infty}^{x_2} \overline{f}(s)ds\right)\mathbb{1}_{\mathbb{R}_-}(x_2) - \left(\int_{x_2}^{\infty} \overline{f}(s)ds\right)\mathbb{1}_{\mathbb{R}_+}(x_2)\right| \\
&\leq 2\|f\|_{L^2_\alpha(Z)}\left(\left\{\int_{-\infty}^{x_2} \rho^{-2\alpha}(y_2)dy_2\right\}^{\frac{1}{2}}\mathbb{1}_{\mathbb{R}_-}(x_2) + \left\{\int_{x_2}^{\infty} \rho^{-2\alpha}(y_2)dy_2\right\}^{\frac{1}{2}}\mathbb{1}_{\mathbb{R}_+}(x_2)\right) \\
&\qquad \leq C\|f\|_{L^2_\alpha(Z)}\rho^{\frac{1}{2}-\alpha}(x_2).
\end{aligned}$$

Notice that since $\alpha > 1/2$, the integrals $\int_{-\infty}^{x_2} \rho^{-2\alpha}(y_2)dy_2$ and $\int_{x_2}^{\infty} \rho^{-2\alpha}(y_2)dy_2$ are defined. Then taking the square, multiplying by $\rho^{2\beta}$ and integrating wrt x

$$\left\|\mathrm{sgn}(x_2) * \overline{f} - \mathrm{sgn}(x_2)\int \overline{f}(s)ds\right\|^2_{L^2_\beta(Z)} \leq C\|f\|^2_{L^2_\alpha(Z)}\int_{\mathbb{R}} \rho^{1-2\alpha+2\beta}(x_2)dx_2,$$

which is bounded, provided that $\beta < \alpha - 1$. Taking into account the polar condition $f\perp\mathbb{R}$ gives then the first statement.

2. We use a duality argument. For any $\varphi \in \mathscr{D}_\#(Z)$, one can write

$$
\begin{aligned}
|\langle \operatorname{sgn} * \overline{f}, \varphi \rangle_{\mathscr{D}'_\#(Z), \mathscr{D}_\#(Z)}| &= |\langle \overline{f}, \operatorname{sgn} * \varphi \rangle_{L^2_\alpha(Z), L^2_{-\alpha}(Z)}| \\
&\leq \|f\|_{L^2_\alpha(Z)} \|\operatorname{sgn} * \varphi\|_{L^2_{-\alpha}(Z)} \leq \|f\|_{L^2_\alpha(Z)} \|\varphi\|_{L^2_{-\alpha+1+\epsilon}(Z) \perp \mathbb{R}},
\end{aligned}
$$

the last inequality being true using the first statement of the theorem provided that $-\alpha + 1 + \epsilon > 1/2$, which is the case if $\alpha \leq \frac{1}{2}$ and $\epsilon > 0$. This proves the second statement.

□

Similarly to Theorem 6.1, thanks to the previous lemma, one gets

Theorem 6.2. *Assume α be a real number such that $\frac{1}{2} < \alpha \leq \frac{3}{2}$. If $f \in \mathscr{D}_\#(Z) \perp \mathbb{R}$ then for any $\epsilon > 0$ one has*

$$
\left\| G_2 * \overline{f} \right\|_{L^2_{\alpha-2-\epsilon}(Z)/\mathbb{R}} \leq C \left\| f \right\|_{L^2_\alpha(Z)}.
$$

The proof of this above theorem is given in D.

We can now summarized Theorem 6.1, Lemma 6.6 and Theorem 6.2.

Theorem 6.3. *Assume $\alpha \in \mathbb{R}$ and $f \in \mathscr{D}_\#(Z) \perp \mathbb{P}'^{\Delta}_{q(0,-\alpha)}$, where $q(0,-\alpha)$ is defined by Proposition 2.2. Then for any $\epsilon > 0$, we have*

$$
\|G_2 * \overline{f}\|_{L^2_{\alpha-2-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}} \leq C \|f\|_{L^2_\alpha(Z)}.
$$

Proof. Let us first observe that the space of the harmonical polynomials of $L^2_{\alpha-2-\epsilon}(Z)$ that only depend on y_2 is the space $\mathbb{P}'^{\Delta}_{q(0,\alpha-2-\epsilon)}$. But since $\epsilon > 0$ is arbitrary, then using (2.14), we have $\mathbb{P}'^{\Delta}_{q(0,\alpha-2-\epsilon)} = \mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}$. Furthermore, using again (2.14), we also have

$$
\mathbb{P}'^{\Delta}_{q(0,-\alpha)} = \begin{cases} \mathbb{R} \text{ if } 1/2 < \alpha \leq 3/2, \\ \\ \mathbb{P}'_1 \text{ if } \alpha > 3/2 \end{cases} \quad \text{and} \quad \mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]} = \begin{cases} \mathbb{R} \text{ if } 1/2 < \alpha \leq 3/2, \\ \\ \{0\} \text{ if } \alpha > 3/2. \end{cases}
$$

Therefore thanks to Theorem 6.1 and Theorem 6.2, the statement is proved for $\alpha > \frac{1}{2}$.

Now as for Lemma 6.6, for the case $\alpha \leq \frac{1}{2}$, we use a duality argument. For any $\varphi \in \mathscr{D}_\#(Z)$, one can write

$$
\begin{aligned}
|\langle G_2 * \overline{f}, \varphi \rangle_{\mathscr{D}'_\#(Z), \mathscr{D}_\#(Z)}| &= |\langle \overline{f}, G_2 * \varphi \rangle_{L^2_\alpha(Z) \perp \mathbb{P}'^{\Delta}_{q(0,-\alpha)}, L^2_{-\alpha}(Z)/\mathbb{P}'^{\Delta}_{q(0,-\alpha)}}| \\
&\leq \|\overline{f}\|_{L^2_\alpha(Z)} \|G_2 * \varphi\|_{L^2_{-\alpha}(Z)/\mathbb{P}'^{\Delta}_{q(0,-\alpha)}} \\
&\leq \|f\|_{L^2_\alpha(Z)} \|\varphi\|_{L^2_{-\alpha+2+\epsilon}(Z) \perp \mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}},
\end{aligned}
$$

the last inequality being true provided that $-\alpha + 2 + \epsilon > \frac{1}{2}$ which is the case if $\alpha \leq \frac{1}{2}$. □

We extend the above statement to data that belong to $L^2_\alpha(Z)$.

Theorem 6.4. *Let $\alpha \in \mathbb{R}$ and recall that $q(0,-\alpha)$ is defined by (2.14). Then for any function $f \in L^2_\alpha(Z) \perp \mathbb{P}'^{\Delta}_{q(0,-\alpha)}$, one defines by density the convolution $G * f$ as a function in $H^1_{\alpha-1-\epsilon,\#}(Z)/\mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}$, for any $\epsilon > 0$, and one has the estimates*

$$\left\| G * f \right\|_{H^1_{\alpha-1-\epsilon,\#}(Z)/\mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}} \leq C \left\| f \right\|_{L^2_\alpha(Z) \perp \mathbb{P}'^{\Delta}_{q(0,-\alpha)}},$$

where the constant C does not depend on the data.

Proof. The result is obtained by density. Proceeding as in the proof of Theorem 6.2, one is able to construct a regular sequence $f_\delta \in \mathscr{D}_\#(Z)$ orthogonal to $\mathbb{P}'^{\Delta}_{q(0,-\alpha)}$ converging strongly to f in the $L^2_\alpha(Z)$ norm. Then Lemma 6.1 and 6.2 provide the decomposition :

$$G * f_\delta = G * (f_\delta - \overline{f}_\delta) + G * \overline{f}_\delta = G_1 * (f_\delta - \overline{f}_\delta) + G_2 * \overline{f}_\delta. \tag{6.37}$$

Then it suffices now to apply Proposition 6.2, Lemma 6.6 and Theorem 6.3 to obtain that

$$\begin{aligned}\left\| G * f_\delta \right\|_{H^1_{\alpha-1-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}} &\leq C \left\| G_1 * (f_\delta - \overline{f}_\delta) \right\|_{H^1_{\alpha-1-\epsilon,\#}(Z)} + \left\| G_2 * \overline{f}_\delta \right\|_{H^1_{\alpha-1-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{3}{2}-\alpha]}} \\ &\leq C \left\| f_\delta \right\|_{L^2_\alpha(Z) \perp \mathbb{P}'^{\Delta}_{q(0,-\alpha)}} \leq C \left\| f \right\|_{L^2_\alpha(Z) \perp \mathbb{P}'^{\Delta}_{q(0,-\alpha)}}.\end{aligned}$$

Then passing to the limit wrt δ extends the convolution to any function $f \in L^2_\alpha(Z) \perp \mathbb{P}_{[\alpha-\frac{1}{2}]}$ □

We generalize the latter convolutions to weak data, namely, to $f \in H^{-1}_{\alpha,\#}(Z)$.

Proposition 6.3. *Let $\alpha \in \mathbb{R}$ and $f \in H^{-1}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(1,-\alpha)}$, where $q(1,-\alpha)$ is defined by (2.13). Then, for any $\epsilon > 0$, we have*

$$\left\| G * f \right\|_{L^2_{\alpha-1-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]}} \leq C \left\| f \right\|_{H^{-1}_{\alpha,\#}(Z)},$$

where C is independent on f.

Proof. The proof is again based on a duality argument. Indeed, thanks to Theorem 6.4, for any $\varphi \in \mathscr{D}_\#(Z)$, we can write

$$\begin{aligned}\langle f, G * \varphi \rangle_{H^{-1}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{q(1,-\alpha)} \times H^1_{-\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{q(1,-\alpha)}} &\leq \left\| f \right\|_{H^{-1}_{\alpha,\#}(Z)} \left\| G * \varphi \right\|_{H^1_{-\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{q(1,-\alpha)}} \\ &\leq C \left\| f \right\|_{H^{-1}_{\alpha,\#}(Z)} \left\| \varphi \right\|_{L^2_{-\alpha+1+\epsilon}(Z) \perp \mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]}},\end{aligned}$$

which proves the claim. □

We are now in a position to give the proof of Theorem 1.2.

Sketch of the proof of Theorem 1.2. Observe first that since we assume that $\alpha \notin \mathbb{Z} + \frac{1}{2}$, then the polynomial spaces $\mathbb{P}'^{\Delta}_{q(1,-\alpha)}$ and $\mathbb{P}'^{\Delta}_{[1/2+\alpha]}$ coincide. We shall now decompose the proof into two steps:

1. In the first step we prove the statement for $m = -1$ which amount to prove that if $\alpha \in \mathbb{R}$ satisfies $\alpha \notin \mathbb{Z} + \frac{1}{2}$, and $f \in H^{-1}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[1/2+\alpha]}$, then $G * f \in H^1_{\alpha,\#}(Z)$ is a solution of the Laplace equation (5.20) unique up to a polynomial of $\mathbb{P}'^{\Delta}_{[1/2-\alpha]}$. Moreover, we have the estimate
$$\|G * f\|_{H^1_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[1/2-\alpha]}} \leq C \|f\|_{H^{-1}_{\alpha,\#}(Z)}.$$
Since $f \in H^{-1}_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[1/2+\alpha]}$, then on the one hand, Proposition 6.3 yields that $G * f$ belongs to $L^2_{\alpha-1-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]}$, for any $\epsilon > 0$, and on the other hand, isomorphism (5.32) shows that there exists a unique u in $H^1_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]} \subset L^2_{\alpha-1-\epsilon}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]}$. Therefore thanks to Proposition 5.1, $u - G * f$ is a polynomial of $\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]} \subset H^1_{\alpha,\#}(Z)$ which in turn shows that $G * f$ belongs to $H^1_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[\frac{1}{2}-\alpha]}$.

2. In the second step we first assume $m \geq 0$, $\alpha \in \mathbb{R}$ satisfying $\alpha \notin \mathbb{Z} + \frac{1}{2}$ and let $f \in H^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[-m-1/2+\alpha]}$. Then it is clear that $f \in H^{-1}_{\alpha-m-1,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[-m-1/2+\alpha]}$. On the one hand, from the first step $G * f$ belongs to the space $H^1_{\alpha-m-1,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]}$. On the other hand, from (5.33), there exists a unique u in $H^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]}$, a subset of $H^1_{\alpha-m-1,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+3/2-\alpha]}$. Then using Proposition 5.1 we have $u - G * f \in \mathbb{P}'^{\Delta}_{[m+3/2-\alpha]}$.
Finally using a duality argument we prove the statement for $m \leq 0$ and $\alpha \in \mathbb{R}$ satisfying $\alpha \notin \mathbb{Z} + \frac{1}{2}$.

□

7 Solutions for data not satisfying any polar condition

We recall here Lemma 4.9 p. 40 [28],

Lemma 7.1. *Let E be a Banach space, F a dense subset in E, M a subspace of finite dimension k of E' and $(f_1, \ldots, f_k)$ a basis in M there exists a sequence $(g_1, \ldots, g_k)$ in F s.t.*
$$< f_i, g_j >_{E',E} = \delta_{ij},$$
and every element $\varphi \in E$ can be decomposed as $\varphi = g_0 + h$ s.t.
$$g_0 = \sum_{i=1}^{k} < h, f_i > g_i, \quad h := \varphi - g_0$$
where $g_0 \in F$ and $h \in E \perp M$.

We now give a result that solves (1.1) when the datum f does not satisfy any polar condition.

Theorem 7.1. *Let $m \in \mathbb{N}$ and $\alpha \in (\frac{1}{2} + m, \infty)$ and $\alpha \notin \mathbb{Z} + \frac{1}{2}$. If $f \in H^m_{\alpha,\#}(Z)$, (f does not satisfy any polar condition) there exists a unique $u \in H^{m+2}_{m+\frac{1}{2}-\delta,\#}(Z)/\mathbb{P}'^{\Delta}_1$ solving (1.1). Moreover there exists $g \in \mathscr{D}_{\#}(Z)$ s.t.*
$$\left\| u - (1 - \chi(|y_2|)) H(y_2, g) \right\|_{H^{m+2}_{\alpha,\#}(Z)} \leq C \left\| f \right\|_{H^m_{\alpha,\#}(Z)}$$
where the constant does not depend on the data nor on u and χ is a cut-off function equal to 1 in the neighborhood of zero and vanishing out of a compact support.

Proof. By Lemma 7.1, one has the decomposition $f = h + g$ where $h \in H^m_{\alpha,\#}(Z) \perp \mathbb{P}'^{\Delta}_{[\alpha-\frac{1}{2}-m]}$ and $g \in \mathscr{D}_\#(Z)$. By (5.33) there exists a unique $u_h \in H^{m+2}_{\alpha,\#}(Z)/\mathbb{P}'^{\Delta}_{[m+\frac{3}{2}-\alpha]}$ solving $-\Delta u_h = h$. On the other hand, by Theorem 6.1 combined with the proof of Lemma 6.6, and Theorem 2, in F, one gets easily, because of the compact support of g, that $u_g = G * g$ satisfies

$$\left\| u_g - (1-\chi(|y_2|)H_0(y_2,g) \right\|_{H^{m+2}_{m+\alpha'-\delta,\#}(Z)} \le C \|g\|_{H^m_{\alpha',\#}(Z)}$$

for all $\alpha' > \frac{3}{2} + m$. If $\alpha > \frac{3}{2} + m$, we choose $\alpha' = \alpha + 2\delta$ otherwise any $\alpha' > \frac{3}{2} + m$ is convenient. This gives finally, setting $u := u_g + u_h$, that

$$\left\| u - (1-\chi(|y_2|)H_0(y_2,g) \right\|_{H^{m+2}_{m+\alpha,\#}(Z)} \le C \|f\|_{H^m_{\alpha,\#}(Z)}.$$

Because $f \in H^m_{m-\frac{1}{2}-\delta,\#}(Z)$ using the isomorphisms already established one concludes that there exists a solution $u_f \in H^{m+2}_{m+\frac{3}{2}-\delta,\#}(Z)$ solving (1.1). Both u_f and u belong to $H^1_{-\frac{3}{2}-\delta,\#}(Z)$ and using isomorphisms results in this latter space by uniqueness one identifies $u \equiv u_f$ modulo polynomials in $\mathbb{P}'^{\Delta}_1$ which ends the proof. $\square$

Appendices

A Proof of Proposition 2.3

For the proof of Proposition 2.3, the ideas of the proof come from [1] and [29]. Let u be in $H^m_{\alpha,\#}(Z)$.

1. We first approximate u by functions with compact support in the y_2 direction. Let $\Phi \in \mathscr{C}^\infty([0,\infty[)$ such that $\Phi(t) = 0$ for $0 \le t \le 1$, $0 \le \Phi(t) \le 1$, for $1 \le t \le 2$ and $\Phi(t) = 1$, for $t \ge 2$. For $\ell \in \mathbb{N}$, we introduce the function Φ_ℓ, defined by

$$\Phi_\ell(t) = \begin{cases} \Phi\left(\dfrac{\ell}{\ln t}\right) & \text{if } t > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{A.38}$$

Note that we have $\Phi_\ell(t) = 1$ if $0 \le t \le e^{\ell/2}$, $0 \le \Phi_\ell(t) \le 1$ if $e^{\ell/2} \le t \le e^\ell$ and $\Phi_\ell(t) = 0$ if $t \ge e^\ell$. Moreover, for all $\ell \ge 2$, $t \in [e^{\ell/2}, e^\ell]$, $\lambda \in \mathbb{N}$, owing that $t \le \sqrt{1+t^2} \le \sqrt{2}\,t$ and $\ln t \le \ln(2+t^2) \le 3\ln t$, we prove that (see [1], Lemma 7.1):

$$\left| \frac{d^\lambda}{dt^\lambda} \Phi\left(\frac{\ell}{\ln t}\right) \right| \le \frac{C}{(1+t^2)^{\lambda/2}\ln(2+t^2)},$$

where C is a constant independent of ℓ. For a.e. $y \in Z$, we set $u_\ell(\boldsymbol{y}) = u(\boldsymbol{y})\Phi_\ell(y_2)$. Then, proceeding as in [1] (Theorem 7.2), one checks easily that u_ℓ belongs to $H^m_{\alpha,\#}(Z)$, has a compact support in the y_2 direction, and that u_ℓ converges to u in $H^m_{\alpha,\#}(Z)$ as ℓ tends to ∞.

Thus, the functions of $H^m_{\alpha,\#}(Z)$ with compact support in the y_2 direction are dense in $H^m_{\alpha,\#}(Z)$ and, we may assume that u has a compact support in the y_2 direction.

2. Let θ be a periodical $\mathscr{D}(\mathbb{R})$-partition of unity and let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence such that $\alpha_j \in \mathscr{D}(\mathbb{R}^2)$, $\alpha_j \geq 0$, $\int_{\mathbb{R}^2} \alpha_j(\boldsymbol{x})d\boldsymbol{x} = 1$ and the support of α_j is included in the closed ball of radius $r_j > 0$ and centered at $(0,0)$ where $r_j \to 0$ as $j \to \infty$. It is well known that as $j \to \infty$, α_j converges in the distributional sense to the Dirac measure. We set $w(\boldsymbol{y}) = \theta(y_1)u(\boldsymbol{y})$. Then w belongs to $H^m(\mathbb{R}^2)$ and has a compact support. Moreover, since $\overline{\omega}\theta = 1$, we have $\overline{\omega}w = \overline{\omega}(\theta u) = (\overline{\omega}\theta)u = u$. We define $\varphi_j = w * \alpha_j$. Then φ_j belongs to $\mathscr{D}(\mathbb{R}^2)$ and converges to w in $H^m(\mathbb{R}^2)$ as j tends to ∞. Let $\psi_j = \overline{\omega}\varphi_j$, then ψ_j belongs to $\mathscr{D}_\#(Z)$ and thanks to Lemma 2.1, ψ_j converges to $\overline{\omega}\, w = u$ in $H^m_{\alpha,\#}(Z)$ as j tends to ∞.

B Proof of Proposition 6.1

We define:

$$G = \lim_{N\to\infty} G_N = \lim_{N\to\infty} \sum_{|k|<N} J_k e^{i\tilde{k}y_1}, \text{ with } J_k = \int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{k}^2 + \tilde{l}^2} dl = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\tilde{l}|y_2|}}{\tilde{k}^2 + \tilde{l}^2} dl.$$

The absolute value in the last right hand side is added as follows:

$$\int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{k}^2 + \tilde{l}^2} dl = \int_0^\infty \frac{e^{i\tilde{l}y_2} + e^{-i\tilde{l}y_2}}{\tilde{k}^2 + \tilde{l}^2} dl = 2\int_0^\infty \frac{\cos(\tilde{l}y_2)}{\tilde{k}^2 + \tilde{l}^2} dl = 2\int_0^\infty \frac{\cos(\tilde{l}|y_2|)}{\tilde{k}^2 + \tilde{l}^2} dl.$$

Proceeding as in example 1 p. 58 [26], one has, extending the integral to the complex plane, that

$$\begin{aligned} J_k &\equiv \lim_{R\to\infty} \oint_{C_R} \frac{e^{i\tilde{z}|y_2|}}{\tilde{k}^2 + \tilde{z}^2} dz = \lim_{R\to\infty} \oint_{C_R} \frac{e^{i\tilde{z}|y_2|}}{(\tilde{z} + i|\tilde{k}|)(\tilde{z} - i|\tilde{k}|)} dz \\ &= \lim_{R\to\infty} \frac{1}{2\pi} \oint_{C_R} \frac{\mathfrak{f}(z)}{z - i|k|} dz = i\mathfrak{f}(i|k|) = \frac{e^{-|\tilde{k}||y_2|}}{2|\tilde{k}|}, \end{aligned}$$

where $C_R := \{z \in \mathbb{C};\ |z| = R \text{ and } \mathfrak{Im}(z) > 0\} \cup ([-R;R] \times \{0\})$ as depicted in fig. 1 and $\mathfrak{f}(z) := e^{i\tilde{z}|y_2|}/(\tilde{z} + i|\tilde{k}|)$, being a holomorphic function inside C_R. The 1D-Fourier transform of the tempered distribution $|y|$ is $-2/\tilde{l}^2$,

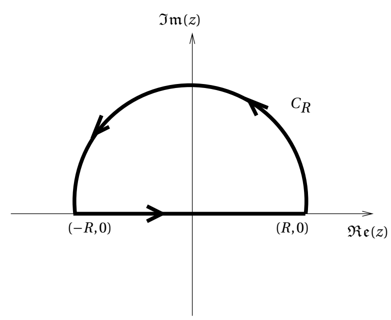

Figure 1: Path of integration in the complex plane

thus one concludes formally that

$$J_0 = \int_{\mathbb{R}} \frac{e^{i\tilde{l}y_2}}{\tilde{l}^2} dl = -\frac{1}{2}|y_2|.$$

The L^2 bound is achieved thanks to the Parseval formula

$$\|G_1\|^2_{L^2(Z)} = \|\mathscr{F}(G_1)\|^2_{\ell^2(\mathbb{Z};L^2(\mathbb{R}))} = \sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}} \frac{1}{(\tilde{k}^2+\tilde{\ell}^2)^2}\,d\ell \le C\sum_{k\in\mathbb{Z}^*}\frac{1}{|\tilde{k}|^3} \le C'.$$

One recalls the expansion in series of the logarithm :

$$\ln(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k}, \quad \forall z\in\mathbb{C} : |z|<1.$$

Thus for $\boldsymbol{y}\neq 0$

$$\begin{aligned} G_1(\boldsymbol{y}) &= -\frac{1}{2\pi}\mathfrak{Re}\ln\left\{1-e^{2\pi(-|y_2|+iy_1)}\right\} = -\frac{1}{2\pi}\ln\left|1-e^{2\pi(-|y_2|+iy_1)}\right| \\ &= -\frac{1}{2\pi}\ln\left\{e^{-\pi|y_2|}\sqrt{2}\left(\cosh(2\pi|y_2|)-\cos(2\pi y_1)\right)^{\frac{1}{2}}\right\} \\ &= \frac{|y_2|}{2} - \frac{1}{4\pi}\ln(2\left(\cosh(2\pi y_2)-\cos(2\pi y_1)\right)), \end{aligned}$$

which gives the desired result for G. □

C Proof of Lemma 6.5

We set $|x| = R$ and let R_0 be a real such that $R_0 > 1$. We define three regions of the real line :

$$D_1 := B(0,R/2), \quad D_2 := B(0,2R)\setminus B(0,R/2), \quad D_3 := \mathbb{R}\setminus B(0,2R)$$

Decomposing the convolution integral in these three parts, one gets :

$$I(x) = \sum_{i=1}^{3} I_i(x), \quad \text{where } I_i(x) = \int_{D_i} e^{-|a||x-y|}\rho(y)^{-b}\,dy.$$

We have two cases to investigate : $R > R_0$ and $R \le R_0$.

1. If $R > R_0$

   (a) On $|y| \le R/2$, $|x-y| \sim R$ so that

   $$I_1(x) \sim e^{-|a|R}\int_{|y|\le\frac{R}{2}}(1+|y|)^{-b}\,dy.$$

   According to the value of b, one has

   $$I_1(x) \le Ce^{-|a|R}\begin{cases} 1 & \text{if } b>1, \\ R & \text{if } 0\le b\le 1, \\ R^{1-b} & \text{if } b<0. \end{cases}$$

   As $R>1$, there holds

   $$e^{-|a|R} \le \frac{([\sigma]+1)!}{(|a|R)^{\sigma}}.$$

   Setting

   $$\sigma(b) := b\,\mathbb{1}_{]1,+\infty[}(b) + (1+b)\,\mathbb{1}_{[0,1]}(b) + \mathbb{1}_{]-\infty,0[}(b),$$

   one recovers the claim in that case.

(b) On $R/2 \leq |y| \leq 2R$, $|y| \sim R$

$$I_2(x) \sim R^{-b} \int_{\frac{R}{2} \leq |y| \leq 2R} e^{-|a||x-y|} \, dy,$$

because $|x-y| \leq 3R$, one has $I_2(x) \sim R^{-b}/|a|$.

(c) On the rest of the line $|y| > 2R$, $|x-y| \sim |y|$, so that

$$I_3(x) \sim \int_{|y|>2R} e^{-|a||y|} |y|^{-b} dy.$$

Two situations occur :

- either $b \leq 0$ then we set $\tilde{b} := [-b] + 1$ and one has

$$\begin{aligned} \int_{|y|>2R} e^{-|a||y|} |y|^{-b} dy &\leq C \frac{e^{-|a|R}}{|a|} \left\{ R^{\tilde{b}} \sum_{p=0}^{\tilde{b}} \frac{\tilde{b}!}{(\tilde{b}-p)!} (|a|R)^{-p} \right\} \\ &\leq C \frac{e^{-|a|R}}{|a|} R^{\tilde{b}} \tilde{b}! \tilde{b} \leq \frac{C(b) R^{-b}}{|a|}. \end{aligned}$$

  In the latter inequality we used decreasing properties of the exponential function in order to fix the correct behavior for large R.

- either $b > 0$ and one has directly that

$$\int_{|y|>2R} e^{-|a||y|} |y|^{-b} dy \leq R^{-b} \int_{|y|>R} e^{-|a||y|} dy.$$

2. Otherwise, if $R < R_0$ then $\rho(x-y) \sim \rho(y)$, and thus

$$I(x) \sim \int_{\mathbb{R}} e^{-|a||y|} \rho(y)^{-b} dy < \frac{C}{|a|}. \quad \square$$

D Proof of Theorem 6.2

As is the proofs of Theorem 6.1 and Lemma 6.6, the proof is essentially 1D. Let us choose $f \in \mathscr{D}_\#(\mathbb{R}) \perp \mathbb{R}$ and a test function $\varphi \in L^2_{2-\alpha+\epsilon}(\mathbb{R}) \perp \mathbb{R}$. We construct $(\varphi_\delta)_\delta \in \mathscr{D}(\mathbb{R}) \perp \mathbb{R}$ s.t. $\varphi_\delta \to \varphi$ in $L^2_{2-\alpha+\epsilon}(\mathbb{R}) \perp \mathbb{R}$: by truncation and smoothing we set $\psi_\delta \in \mathscr{D}_\#(Z)$ s.t. $\psi_\delta \to \varphi$ in $L^2_{2-\alpha+\epsilon}(\mathbb{R}) \perp \mathbb{R}$ strongly. Then we choose an arbitrary function $\breve{\psi} \in \mathscr{D}_\#(Z)$ s.t. $\int_{\mathbb{R}} \breve{\psi} \, dy = 1$ and set

$$\varphi_\delta(y) := \psi_\delta(y) - \left(\int_{\mathbb{R}} \psi_\delta \, dy \right) \breve{\psi}.$$

It is easy to check that $\varphi_\delta \perp \mathbb{R}$ and that for every ϵ_1 there exists δ_0 s.t. $\delta < \delta_0$,

$$\begin{aligned} \|\varphi_\delta - \varphi\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})} &\leq \|\psi_\delta - \varphi\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})} + \left(\left| \int_{-\infty}^{-\frac{1}{\delta}} \varphi \, dy \right| + \left| \int_{\frac{1}{\delta}}^{\infty} \varphi \, dy \right| \right) \|\breve{\psi}\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})} \\ &\leq \epsilon_1/2 + 2\delta^{\frac{3}{2}+\epsilon-\alpha} \|\varphi\|_{L^2_{2-\alpha+\epsilon}} \|\breve{\psi}\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})} \leq \epsilon_1 \end{aligned}$$

where we used that

$$\int_{\mathbb{R}} \psi_\delta \, dy = \int_{\mathbb{R}} \left(\varphi \mathbb{1}_{]-\frac{1}{\delta}, \frac{1}{\delta}[} \right) * \Theta_\delta \, dy = \int_{\mathbb{R}} \left(\varphi \mathbb{1}_{]-\frac{1}{\delta}, \frac{1}{\delta}[} \right) dz \int_{\mathbb{R}} \Theta_\delta \, dy$$

Θ_δ being the standard mollifier whose mean value is one, presented in (ii) part of the proof of Proposition 2.3, we used also the fact that

$$\int_{]-\frac{1}{\delta},\frac{1}{\delta}[}\varphi dz = -\left(\int_{]-\infty,-\frac{1}{\delta}[}\varphi dz + \int_{]\frac{1}{\delta},\infty[}\varphi dz\right).$$

due to the orthogonality condition $\int_{\mathbb{R}}\varphi dy = 0$. Then $|x| * \varphi_\delta$ is infinitely differentiable and one can apply the Taylor expansion with the integral rest :

$$\begin{aligned}
&\int_{\mathbb{R}}(|x| * \varphi_\delta)(x) f(x)dx \\
&= \int_{\mathbb{R}}\left\{(|x| * \varphi_\delta)(0) + \int_0^x (\varphi_\delta * \mathrm{sgn})(s)ds\right\} f(x)dx \equiv \int_{\mathbb{R}} f(x)\int_0^x (\varphi_\delta * \mathrm{sgn})(s)dsdx.
\end{aligned}$$

Note that the regularization of φ plays here a crucial role since $|x| * \varphi(0) = \int_{\mathbb{R}}|x\text{Âă}|\varphi(x)dx$ does not make sense if $\varphi \in L^2_{2-\alpha+\epsilon}(\mathbb{R})$ for $\alpha \in]\frac{1}{2},\frac{3}{2}]$. Proceeding as in the proof of Lemma 6.6, one has

$$\left|\int_0^x \varphi_\delta * \mathrm{sgn}(s)\, ds\right| \leq \|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(Z)}\rho^{\alpha-\frac{1}{2}-\epsilon}(x),$$

leading to

$$\left\|\int_0^x (\varphi_\delta * \mathrm{sgn})(s)ds\right\|_{L^2_{-\alpha}(\mathbb{R})} \leq C\|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})},$$

with $\alpha < \frac{3}{2} + \epsilon$. Thus one has

$$\left|\int_{\mathbb{R}}(|x| * \varphi_\delta)(x) f(x)dxdy\right| \leq C\|f\|_{L^2_\alpha(\mathbb{R})}\|\varphi_\delta\|_{L^2_{2-\alpha+\epsilon}(\mathbb{R})}.$$

Moreover, one needs that $\mathbb{R} \in L^2_{-\alpha}(\mathbb{R})$, which is true if $\alpha > \frac{1}{2}$. It is not difficult to prove by similar arguments that

$$\lim_{\delta\to 0}\int_0^x (\varphi_\delta * \mathrm{sgn})(s)ds = \int_0^x (\varphi * \mathrm{sgn})(s)ds$$

strongly in the $L^2_{-\alpha}(\mathbb{R})$ topology. As in the proof of Lemma 6.6, by Fubini,

$$\int_0^x (\mathrm{sgn} * \varphi)(s)ds = 2\int_0^x \left\{\int_{-\infty}^s \varphi(t)dt\mathbb{1}_{s<0}(s) - \int_s^\infty \varphi(t)dt\mathbb{1}_{s>0}(s)\right\}ds =: 2\int_0^x \mathfrak{g}(s)ds.$$

As $\mathfrak{g}$ is a C^1 function on any compact set in $\mathbb{R}$, one can integrate by parts on $(0,x)$:

$$\int_0^x \mathfrak{g}(s)ds = [\mathfrak{g}(s)s]_0^x - \int_0^x \mathfrak{g}'(s)\, s\, ds.$$

Using this in the right hand side of the previous limit, one writes

$$\begin{aligned}
J &:= \int_{\mathbb{R}} f(x)\int_0^x (\varphi * \mathrm{sgn})(s)dsdx = 2\int_{\mathbb{R}} f(x)\left\{\mathfrak{g}(x)x - \int_0^x \mathfrak{g}'(s)\, s\, ds\right\}dx \\
&= 2\left\{\int_{\mathbb{R}_-} f(x)\, x\int_{-\infty}^x \varphi(t)dtdx - \int_{\mathbb{R}_+} f(x)x\int_x^\infty \varphi(t)dtdx - \int_{\mathbb{R}} f(x)\int_0^x \varphi(s)\, s\, ds\right\} \\
&= 2\int_{\mathbb{R}} f(x)\, x\int_{-\infty}^x \varphi(t)dtdx - 2\int_{\mathbb{R}} f(x)\int_0^x \varphi(s)\, s\, dsdx =: A - B.
\end{aligned}$$

By Hölder estimates the integrals above are well defined. Fubini's theorem allows then to switch integration order. Using the orthogonality condition on φ and on f, one may easily show that

$$\begin{cases} A = 2\int_{\mathbb{R}_x} f(x)\,x\left(\int_{-\infty}^{x} \varphi(t)\,dt\right)dx = \int_{\mathbb{R}} f(x)x\left\{\int_{-\infty}^{x}\varphi(s)ds - \int_{x}^{\infty}\varphi(s)ds\right\}dx \\ \quad = \int_{\mathbb{R}_s}\varphi(s)\left\{\int_{s}^{\infty} xf(x)dx - \int_{-\infty}^{s} xf(x)dx\right\}ds, \\ B = 2\left\{\int_{\mathbb{R}_+} f(x)\int_0^x s\varphi(s)\,ds\,dx - \int_{\mathbb{R}_-} f(x)\int_x^0 s\varphi(s)\,ds\,dx\right\} \\ \quad = 2\left\{\int_{\mathbb{R}_+} s\varphi(s)\int_s^{\infty} f(x)dxds - \int_{\mathbb{R}_-} s\varphi(s)\int_{-\infty}^{s} f(x)dxds\right\} \\ \quad = -2\int_{\mathbb{R}} s\varphi(s)\int_{-\infty}^{s} f(x)dxds = \int_{\mathbb{R}} s\varphi(s)\left\{\int_s^{\infty} f(x)dxds - \int_{-\infty}^{s} f(x)dx\right\}ds. \end{cases}$$

These computations give :

$$\begin{aligned} A - B &= \int_{\mathbb{R}_s}\varphi(s)\left\{\int_s^{\infty}(x-s)f(x)dx - \int_{-\infty}^{s}(x-s)f(x)dx\right\}ds \\ &= \int_{\mathbb{R}}\varphi(s)\int_{\mathbb{R}}|x-s|f(x)dxds = \int_{\mathbb{R}}\varphi(s)(|x|*f)(s)ds. \end{aligned}$$

And because

$$\inf_{\lambda\in\mathbb{R}}\left\||x|*f+\lambda\right\|_{L^2_{\alpha-(2+\epsilon)}(\mathbb{R})} = \sup_{\varphi\in L^2_{(2+\epsilon)-\alpha}(\mathbb{R})\perp\mathbb{R}}\frac{(|x|*f,\varphi)}{\|\varphi\|_{L^2_{(2+\epsilon)-\alpha}(\mathbb{R})}},$$

the final claim follows.

E Proposition 5.2

We define $\mathcal{V} := \{\mathbf{u}\in\mathcal{D}_\#(Z) \text{ s.t. } \operatorname{div}\mathbf{u} = 0\}$, $\nabla^\perp := (\partial_{y_2}, -\partial_{y_1})^T$ a subspace of $H^1_{\alpha,\#}(Z)$ denoted $\mathbf{V}_\alpha := \{\mathbf{u}\in L^2_\alpha(Z) \text{ s.t. } \operatorname{div}\mathbf{u} = 0\}$.

Proposition 1. *Let $\alpha = \frac{1}{2}$ given a function $\mathbf{u}\in\mathbf{V}_\alpha$ then there exists a function $\varphi\in H^1_{\frac{1}{2},\#}(Z)$ s.t.*

$$\nabla^\perp\varphi = \mathbf{u}, \quad a.e\ \boldsymbol{y}\in Z.$$

Proof. We use the isomorphisms of the Laplace operator of Theorem 5.1. A simple computation gives that $\operatorname{rot}\mathbf{u}\in H^{-1}_{\frac{1}{2},\#}(Z)\perp\mathbb{R}$ thus : looking for φ s.t.

$$\Delta\varphi = \operatorname{rot}\mathbf{u}, \quad \boldsymbol{y}\in Z.$$

there exists a unique $\varphi\in H^1_{\frac{1}{2},\#}(Z)/\mathbb{R}$ solving this problem. Then applying the operator $\nabla^\perp$ to this equation one gets:

$$\Delta(\nabla^\perp\varphi - \mathbf{u}) = 0$$

which by Proposition 5.1 implies the claim. □

Proposition 2. *If $\alpha = \frac{1}{2}$, the set $\mathscr{V}$ is dense in $\mathbf{V}_\alpha$.*

Proof. We take $\mathbf{u} \in \mathbf{V}_\alpha$ and by Proposition 1 we construct $\varphi \in H^1_{\frac{1}{2},\#}(Z)/\mathbb{R}$ satisfying $\nabla^\perp \varphi = \mathbf{u}$. We approximate it by a sequence $\varphi_j \in \mathscr{D}_\#(Z)$, as $\nabla^\perp$ is a continuous operator one has

$$\nabla^\perp \varphi_j \to \nabla^\perp \varphi, \quad \text{strongly in } L^2_\alpha(Z).$$

this ends the proof. □

We are now ready to prove the proposition. Let $u \in \mathscr{D}'_\#(Z)$ such that $\nabla u \in L^2_{-\frac{1}{2}}(Z)$. Then for any $\varphi \in \mathscr{D}_\#(Z)$, we have

$$\langle \nabla u, \varphi \rangle = -\langle u, \operatorname{div} \varphi \rangle,$$

which implies that ∇u is orthogonal to $\mathscr{V}$. Due to the density of $\mathscr{V}$ in $\mathbf{V}_{\frac{1}{2}}$, we deduce that ∇u is also orthogonal to $\mathbf{V}_{\frac{1}{2}}$. Thanks to Proposition 3.1, there exists $w \in H^1_{-\frac{1}{2},\#}(Z)$, such that $\nabla w = \nabla u$. Therefore there exists $K \in \mathbb{R}$ such that $w = u + K \in H^1_{-\frac{1}{2},\#}(Z)$. Now since constants belong to $H^1_{-\frac{1}{2},\#}(Z)$, we deduce that $u \in H^1_{-\frac{1}{2},\#}(Z)$. The estimate follows from Proposition 3.1.

F Weighted estimates for higher order derivatives of the convolution with G

Lemma 1. *If $|y_2| > 4$ then for all multi-index $\boldsymbol{\gamma}$ s.t. $|\boldsymbol{\gamma}| = 2$ one has*

$$|\partial^{\boldsymbol{\gamma}} G(\boldsymbol{y})| \leq C e^{-2\pi |y_2|}$$

The constant C does not depend on y_2.

Proof. A simple computation of second order derivatives gives :

$$|\partial^{\boldsymbol{\gamma}} G(\boldsymbol{y})| \leq \frac{C}{\cosh(2\pi y_2)\left(1 - \frac{\cos(2\pi y_1)}{\cosh(2\pi y_2)}\right)^2}$$

and the constant C does not depend on y_2. If $|y_2| > 4 > \ln(4)/2\pi$ then $\cosh(2\pi y_2) > 2$ and thus

$$-\frac{1}{2} < \frac{\cos(2\pi y_1)}{\cosh(2\pi y_2)} < \frac{1}{2}$$

thus

$$|\partial^{\boldsymbol{\gamma}} G(\boldsymbol{y})| \leq C \cosh(2\pi y_2)^{-1} \frac{1}{\left(1 - \frac{1}{2}\right)^2} \leq C' e^{-2\pi |y_2|}.$$

□

Lemma 2. *If $g \in \mathscr{D}_\#(Z)$ then if we denote $K := \operatorname{supp}(g)$ and one denotes $\ell := \sup_{\boldsymbol{y} \in K} |y_2|$ one has that*

$$\left|(\partial^{\boldsymbol{\gamma}} G * g)(\boldsymbol{y})\right| \leq C \|g\|_{L^1(Z)} e^{-\pi |y_2|}, \quad \forall \boldsymbol{y} \text{ s.t. } |y_2| > \max(2\ell, 2), \forall \boldsymbol{\gamma} \in \mathbb{N}^2 \text{ s.t. } |\boldsymbol{\gamma}| = 2$$

Proof. Using that if $\boldsymbol{z}$ s.t. $|z_2| > 2\ell$ then

$$|z_2 - y_2| > \frac{1}{2}|z_2| > \ell$$

thus applying the previous lemma, one has then also that

$$|(\partial^{\boldsymbol{\gamma}} G)(\boldsymbol{z} - \boldsymbol{y})| \leq C e^{-2\pi|z_2 - y_2|} \leq C e^{-\pi|z_2|}, \quad \forall \boldsymbol{z} \text{ s.t. } |z_2| > 2\ell.$$

Now multiplying $(\partial^{\boldsymbol{\gamma}} G)(\boldsymbol{z} - \boldsymbol{y})$ by $g(\boldsymbol{y})$ and integrating then with respect to $\boldsymbol{y}$ on K, one has the claim. $\square$

Theorem 1. *If $g \in \mathscr{D}_{\#}(Z)$ then $\partial^{\boldsymbol{\gamma}} G * g \in L^2_\alpha(Z)$ for any $\alpha \in \mathbb{R}$, and one has*

$$\left\| \partial^{\boldsymbol{\gamma}} G * g \right\|_{L^2_\alpha(Z)} \leq C \left\| g \right\|_{L^2_\alpha(Z)}, \quad \forall \boldsymbol{\gamma} \in \mathbb{N}^2 \text{ s.t. } |\boldsymbol{\gamma}| = 2.$$

where the constant C does not depend on

Proof. As g is in $\mathscr{D}_{\#}(Z)$ it belongs to $H^m_{\alpha,\#}(Z)$ for all $m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Applying Theorem 1.2, one obtains that $G * g$ is $H^2_{\text{loc},\#}(Z)$ which implies that on every compact set K' one has

$$\left\| \partial^{\boldsymbol{\gamma}} G * g \right\|_{L^2_\alpha(K')} \leq C \left\| g \right\|_{L^2_\alpha(Z)}.$$

Using then Lemma 2, one has that

$$\left\| \partial^{\boldsymbol{\gamma}} G * g \right\|_{L^2_\alpha(Z \setminus K')} \leq C \left\| g \right\|_{L^1(Z)}$$

If K' is chosen big enough. As g is compactly supported, all weighted norms are equivalent, which ends the proof. $\square$

By similar arguments one can prove as well :

Theorem 2. *If $g \in \mathscr{D}_{\#}(Z)$ then $\partial^{\boldsymbol{\gamma}} G * g \in L^2_\alpha(Z)$ for any $\alpha \in \mathbb{R}$ and any multi-index $\boldsymbol{\gamma}$ s.t. $|\boldsymbol{\gamma}| \geq 2$, and one has*

$$\left\| \partial^{\boldsymbol{\gamma}} G * g \right\|_{L^2_\alpha(Z)} \leq C \left\| g \right\|_{L^2_\alpha(Z)}.$$

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